Physics 786, Fall 2018
Problem Set 7, Due Wednesday, October 31, 2018.

Final Paper Assignment - Due last day of class

For your final paper, you should identify and describe either a theoretical aspect of or extension of general relativity, or an experiment or application which either makes use of general relativity or provides a test of general relativity. You should explain in detail those aspects of general relativity which are relevant. Example topics include gravity in extra dimensions, torsion, curvature perturbations during inflation, the cosmic microwave background, Gravity Probe B, LIGO, Planck, BICEP.... As a guideline, aim for ten double-spaced pages.

1. *Vacuum Solutions in Three Dimensions*

In this problem we will look for static, isotropic black-holes in three spacetime dimensions.

Assume a metric of the form

$$ds^2 = -e^{2\phi(r)}dt^2 + e^{2\lambda(r)}dr^2 + r^2 d\theta^2.$$

a) Calculate the nonvanishing Christoffel symbols.

b) Calculate the components of the Ricci tensor R_{tt}, R_{rr}, and $R_{\theta\theta}$. The other components of $R_{\mu\nu}$ vanish.

c) Solve the vacuum Einstein equations $R_{\mu\nu} = 0$, assuming that the metric approaches the flat metric as $r \to \infty$. Are there any nontrivial black-hole solutions?

2. *Abandon hope, all ye who enter here.*

A massive test particle is at $r = r_0 < 2GM$ at $t = 0$. Assume that the metric inside the horizon takes the same Schwarzschild form as outside the horizon.

a) Show that

$$\left|\frac{dr}{d\tau}\right| \geq \sqrt{\frac{2GM}{r} - 1}.$$

b) Show that the test particle necessarily reaches $r = 0$ in finite proper time.

3. *The Kerr spacetime*

This problem is required for students registered for graduate credit, but is not required for students registered for undergraduate credit.

In this problem you will consider the rotating black hole metric, discovered by Roy Kerr in 1963. The metric can be written in Boyer-Lindquist form,

$$\begin{aligned} ds^2 \;=\; & -\left[1 - \frac{2GMr}{r^2 + a^2\cos^2\theta}\right] dt^2 - \frac{4GMr\,a\sin^2\theta}{r^2 + a^2\cos^2\theta} dt\,d\phi \\ & + \left[\frac{r^2 + a^2\cos^2\theta}{r^2 - 2GMr + a^2}\right] dr^2 + (r^2 + a^2\cos^2\theta) d\theta^2 \\ & + \left[r^2 + a^2 + \frac{2GMr\,a^2\sin^2\theta}{r^2 + a^2\cos^2\theta}\right] \sin^2\theta\, d\phi^2, \end{aligned}$$

where a is a constant related to the angular velocity of the black hole if $M \neq 0$.

If $M = 0$ then the metric reduces to the Minkowski metric in oblate spheroid coordinates, related to usual Cartesian coordinates by

$$\begin{aligned} x &= \sqrt{r^2 + a^2}\sin\theta\,\cos\phi \\ y &= \sqrt{r^2 + a^2}\sin\theta\,\sin\phi \\ z &= r\cos\theta. \end{aligned}$$

a) Show that if a=0 then the metric reduces to the Schwarzschild metric.

b) For what relations between r and θ are components of the metric divergent? There should be several such relations.

c) The singularity at $r = 0$ with $\theta = \pi/2$ is a curvature singularity. The other metric singularities correspond to the horizons. What is the shape of the curvature singularity in terms of the coordinates x, y and z as defined above?

d) Suppose an observer uses a jet pack to stand still, with constant r, θ and ϕ in the Boyer-Lindquist coordinates. The tangent vector of the observer is $v^\mu \equiv dX^\mu/dt = (1, 0, 0, 0)$. This describes a physical tangent vector only if v^μ is timelike, *i.e.* $v^\mu v^\nu g_{\mu\nu} < 0$. Find the relation(s) between r and θ such that $v^\mu v^\nu g_{\mu\nu} = 0$.

These regions are called ergosurfaces. In the region (called the ergosphere) outside both horizons and inside the outer ergosurface, observers cannot stand still, but are dragged around the black hole. Roger Penrose suggested that energy could be taken from a rotating black hole by sending objects through this region, with part of the objects falling into the black hole and the other parts propelled away from the black hole.

You should compare your results with Fig. 1 in Matt Visser's review at https://arxiv.org/pdf/0706.0622.pdf.