Physics 786, Fall 2018
Problem Set 4 Due Wednesday, October 3, 2018.

1. *Geodesics on the 2-sphere*

In spherical coordinates, the length element on the 2-sphere of radius R takes the form

$$ds^2 = R^2 \left(d\theta^2 + \sin^2\theta \, d\phi^2 \right) .$$

a) With $x^1 = \theta$ and $x^2 = \phi$, the metric $g_{ij} = g_{ji}$ is defined such that $ds^2 = g_{ij} dx^i dx^j$, summed over i and j. What are the components of g_{ij}, written as a 2×2 matrix?

b) Find the nonvanishing components of the connection

$$\Gamma^i_{jk} = \frac{1}{2} g^{im} \left(\frac{\partial g_{mj}}{\partial x^k} + \frac{\partial g_{mk}}{\partial x^j} - \frac{\partial g_{jk}}{\partial x^m} \right) .$$

c) Consider a path parametrized by a parameter t. The paths of shortest distance satisfy the geodesic equation:

$$\frac{d^2 x^i}{dt^2} + \Gamma^i_{jk} \frac{dx^j}{dt} \frac{dx^k}{dt} = 0.$$

Show that arcs along the equator $\theta = \pi/2$ are geodesics on the 2-sphere.

2. *Geometry of the paraboloid*

Consider the 2-dimensional paraboloid described by $z = x^2 + y^2$ embedded in 3-dimensional Euclidean space with Cartesian coordinates $x,\ y,\ z$.

a) What are the components of the metric on the paraboloid described by coordinates x and y?

b) Change variables to $r,\ \theta$, with $x = r\cos\theta$, $y = r\sin\theta$. What are the components of the metric on the paraboloid in these coordinates?

c) Calculate the Christoffel symbols in the r, θ coordinates.

3. *Coordinate transformation of the Christoffel symbols*

Given a metric tensor $g_{\mu\nu}(x)$, consider the coordinate transformation $x^\mu \to x'^\mu(x)$. How does the Christoffel symbol $\Gamma^\mu_{\nu\lambda}$ transform under this coordinate transformation?

Are the Christoffel symbols the components of a tensor under general coordinate transformations?