1. **Geodesics on the 2-sphere**

In spherical coordinates, the length element on the 2-sphere of radius $R$ takes the form

\[ ds^2 = R^2 \left( d\theta^2 + \sin^2 \theta \, d\phi^2 \right). \]

a) With $x^1 = \theta$ and $x^2 = \phi$, the metric $g_{ij} = g_{ji}$ is defined such that $ds^2 = g_{ij}dx^i dx^j$, summed over $i$ and $j$. What are the components of $g_{ij}$, written as a $2 \times 2$ matrix?

b) Find the nonvanishing components of the connection

\[ \Gamma^i_{jk} = \frac{1}{2} g^{im} \left( \frac{\partial g_{mj}}{\partial x^k} + \frac{\partial g_{mk}}{\partial x^j} - \frac{\partial g_{jk}}{\partial x^m} \right). \]

c) Consider a path parametrized by a parameter $t$. The paths of shortest distance satisfy the geodesic equation:

\[ \frac{d^2 x^i}{dt^2} + \Gamma^i_{jk} \frac{dx^j}{dt} \frac{dx^k}{dt} = 0. \]

Show that arcs along the equator $\theta = \pi/2$ are geodesics on the 2-sphere.

2. **Geometry of the paraboloid**

Consider the 2-dimensional paraboloid described by $z = x^2 + y^2$ embedded in 3-dimensional Euclidean space with Cartesian coordinates $x$, $y$, $z$.

a) What are the components of the metric on the paraboloid described by coordinates $x$ and $y$?

b) Change variables to $r$, $\theta$, with $x = r \cos \theta$, $y = r \sin \theta$. What are the components of the metric on the paraboloid in these coordinates?

c) Calculate the Christoffel symbols in the $r$, $\theta$ coordinates.
3. **Coordinate transformation of the Christoffel symbols**

Given a metric tensor $g_{\mu\nu}(x)$, consider the coordinate transformation $x^\mu \rightarrow x'^\mu(x)$. How does the Christoffel symbol $\Gamma^\mu_{\nu\lambda}$ transform under this coordinate transformation?

Are the Christoffel symbols the components of a tensor under general coordinate transformations?