**Physics 786, Fall 2018**
**Problem Set 3** Due Wednesday, September 26, 2018.

1. *Electromagnetic Waves*

We have seen several analogies between electromagnetism and general relativity. Here we will consider electromagnetic waves in a relativistic framework.

In terms of the field strength tensor $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$, half of the source-free Maxwell's equations take the form

$$\partial_\mu F^{\mu\nu} = 0.$$

a) Show that $F_{\mu\nu}$ is invariant under gauge transformations $A_\mu(x) \rightarrow A_\mu(x) + \partial_\mu f(x)$ for any function $f(x)$.

b) Show that a gauge transformation can generically be chosen to enforce the Lorenz gauge condition, $\partial_\mu A^\mu = 0$. What is the residual gauge freedom that remains after the Lorenz gauge condition is imposed?

c) In Lorenz gauge, Maxwell's equations reduce to the wave equation for each component of $A^\mu$. Considering part (b), how many propagating degrees of freedom of the massless vector field are there?

d) Assume a plane-wave solution of the form

$$A^\mu(x) = \epsilon^\mu \, e^{ik\cdot x} + \epsilon^{\mu\,*} \, e^{-ik\cdot x}.$$

What conditions on $k^\mu$ and $\epsilon^\mu$ follow from the wave equation and the Lorenz gauge condition?

e) With the choice $k^\mu = (k,0,0,k)^\mu$, show that the polarization vectors $\epsilon^\mu_{(1)} = (0,1,0,0)^\mu$, $\epsilon^\mu_{(2)} = (0,0,1,0)^\mu$ and $\epsilon^\mu_{(3)} = (1,0,0,1)^\mu$ satisfy the Lorenz gauge condition. Explain why the solution with $\epsilon_{(3)}$ is unphysical.

f) By considering a rotation about the direction of motion, find the helicities of the plane waves with polarization vectors $\epsilon^\mu_\pm \equiv \epsilon^\mu_{(1)} \pm i\epsilon^\mu_{(2)}$ and $\epsilon_{(3)}$. What are the helicities of the physical solutions?