Phys. 786 F'18 Problem Set 10 Solutions

1. $$\dot{R}^2 = \frac{8\pi G_N}{3}\rho R^2 + \frac{\Lambda}{3}R^2 - \underset{k=1}{1}$$

Suppose $\dot{R}(\underset{\text{today}}{t_0}) > 0$ and $\rho > 0$.

If $\Lambda > \frac{3}{R(t_0)^2}$ then as R increases $\frac{\Lambda}{3}R^2 - 1 > 0 \;\forall\, t > t_0$.

Hence $\dot{R} > \sqrt{\left(\frac{\Lambda}{3}R^2 - 1\right)} > 0 \quad \forall\, t > t_0$.

In that case, the $k=+1$ universe expands forever.

2. $$z = \frac{\lambda_{rec} - \lambda_{emit}}{\lambda_{emit}} = \frac{R(t_{rec}) - R(t_{emit})}{R(t_{emit})}$$

$$\approx \frac{\left(\cancel{R(t_{emit})} + \dot{R}(t_{emit})\,(t_{rec} - t_{emit})\right) - \cancel{R(t_{emit})}}{R(t_{emit})}$$

$$= \left.\frac{\dot{R}}{R}\right|_{t_{emit}} (t_{rec} - t_{emit})$$

$$\boxed{z \approx H(t_{emit}) \cdot \frac{r}{c}}$$, using $r \approx c\,\Delta t$ and $H = \dot{R}/R$.

3. We will use a Newtonian approximation for the binary black hole orbits, assuming that properties of the orbit (R, T, etc.) change only a small amount during each orbit. As we will see, this is a bad approximation for the system in question.

a) The power radiated by gravitational radiation for a binary star system w/ equal mass stars in circular orbit is

$$P = \frac{2}{5}\frac{G^4 M^5}{R^5 c^5}.$$

$$M = 30 \times (2\times 10^{30}\ \text{kg})$$
$$R = 1.4 \times 10^5\ \text{m}$$

$$P = \frac{2}{5}\frac{(6.67\times 10^{-11}\ \text{Nm}^2/\text{kg}^2)^4\,(6\times 10^{31}\ \text{kg})^5}{(1.4\times 10^5\ \text{m})^5\,(3\times 10^8\ \text{m/s})^5}$$

$$= \boxed{4.7\times 10^{49}\ \text{Watts}}$$ (That's a lot of light bulbs!)

b) Newtonian orbit: $\frac{GM^2}{(2R)^2} = M\left(\frac{2\pi}{T}\right)^2 R$

$$\rightarrow T = \frac{4\pi R^{3/2}}{(GM)^{1/2}}$$

Total energy: $E = \underbrace{-\frac{GM^2}{2R}}_{\text{potential}} + \underbrace{2\cdot\frac{M}{2}\left(\frac{2\pi}{T}\right)^2 R^2}_{\text{kinetic}}$

$$= -\frac{GM^2}{4R}$$

R changes in time due to power radiated.

G, M = constant.

$\frac{T^2}{R^3} \sim \text{const} \leftrightarrow T^2 E^3 = \text{const}.$

$$\frac{d}{dt}: 2TE^3 \frac{dT}{dt} + 3T^2E^2\frac{dE}{dt} = 0.$$

Change in period per orbit: $\frac{dT}{dt} \times (\text{time of orbit}) = T\frac{dT}{dt}$

assume $\frac{dT}{dt}$ is roughly constant over orbit. — Newtonian approx. (will be a bad approx. here)

$$T\frac{dT}{dt} = -\frac{3}{2}T^2E^{-1}\frac{dE}{dt} = \frac{192\pi^2}{2}\frac{R^4}{G^2M^3}\frac{dE}{dt}$$

$$= -\frac{192\pi^2}{2}\frac{R^4}{G^2M^3}P$$

$$= -\frac{192\pi^2}{5}\frac{(GM)^2}{Rc^5}$$

$$= \boxed{-0.018\,s}$$

Note that $|\Delta T| > T$, so the Newtonian approximation is not valid for this orbit. However, the calculation indicates that the binary black hole orbit decays quickly once the black holes are so close to one another.