

Wednesday 10.27

Plane wave solutions

Assume $T_{\mu\nu} = 0$.

$$(*) \begin{cases} \partial_\alpha \partial^\alpha h_{\mu\nu} = 0 & \text{linearized Einstein equations in harmonic gauge} \\ \partial_\mu h^\mu{}_\nu = \frac{1}{2} \partial_\nu h & \text{harmonic gauge conditions.} \end{cases}$$

We look for plane wave solutions to the set of equations (*).

Plane wave solutions have the form

$$h_{\mu\nu}(x) = \epsilon_{\mu\nu} \exp(i k \cdot x) + \epsilon_{\mu\nu}^* \exp(-i k \cdot x)$$

where $k \cdot x \equiv k_\mu x^\mu = k^\mu x_\mu$, $\epsilon_{\mu\nu} \equiv$ polarization tensor, can be complex.

$$h_{\mu\nu} = h_{\nu\mu} \Leftrightarrow \boxed{\epsilon_{\mu\nu} = \epsilon_{\nu\mu}}$$

Consider derivatives of $\exp(i k \cdot x)$:

$$\begin{aligned} \partial_\alpha \exp(i k_\mu x^\mu) &= \frac{\partial}{\partial x^\alpha} \exp(i k_\mu x^\mu) \\ &= i k_\mu \frac{\partial x^\mu}{\partial x^\alpha} \exp(i k_\mu x^\mu) \\ &= i k_\mu \delta^\mu_\alpha \exp(i k \cdot x) \\ &= i k_\alpha \exp(i k \cdot x) \end{aligned}$$

Similarly, $\partial_\alpha \partial^\alpha \exp(i k \cdot x) = -k_\alpha k^\alpha \exp(i k \cdot x)$.

Then the equations (*) imply

$$\begin{cases} -k_\alpha k^\alpha h_{\mu\nu} = 0 \Rightarrow \boxed{-k_\alpha k^\alpha = 0} \text{ if } h_{\mu\nu} \neq 0 \\ k_\mu \epsilon^\mu{}_\nu = \frac{1}{2} k_\nu \epsilon^\mu{}_\mu \end{cases}$$

Consider the gauge transformation $h_{\mu\nu} \rightarrow h_{\mu\nu} + \partial_\mu \xi_\nu + \partial_\nu \xi_\mu$

$$\text{with } \xi^\mu(x) = -i \epsilon^\mu \exp(i k \cdot x) + i \epsilon^{\mu*} \exp(-i k \cdot x).$$

This is equivalent to $\epsilon_{\mu\nu} \rightarrow \epsilon_{\mu\nu} + k_\mu \epsilon_\nu + k_\nu \epsilon_\mu$.

Under this class of gauge transformations,

$$K_m E^m_\nu \rightarrow K_m E^m_\nu + \frac{K_m K^\mu}{0} E_\nu + \underbrace{(K_m E^m)}_{K \cdot E} K_\nu$$

$$\begin{aligned} \frac{1}{2} K_\nu E^m_m &\rightarrow \frac{1}{2} K_\nu E^m_m + \frac{1}{2} \cdot 2 K_\nu (K^m E_m) \\ &= \frac{1}{2} K_\nu E + K_\nu (K \cdot E) \end{aligned}$$

$$0 = \underbrace{K_m E^m_\nu - \frac{1}{2} K_\nu E^m_m}_{\text{harmonic gauge condition}} \rightarrow K_m E^m_\nu - \frac{1}{2} K_\nu E^m_m$$

Number of independent solutions:

For each K^M satisfying $K_m K^m = 0$,

$E_{\mu\nu}$ — symmetric 4×4 matrix	10 components
harmonic gauge condition	-4
remaining gauge freedom	-4

2 independent polarizations

— like EM!

Example: Wave traveling in x^3 -direction.

$$K^1 = K^2 = 0, \quad K^3 = K^0 \equiv K > 0.$$

$$\text{Harmonic conditions: } \begin{cases} K^3 E_{31} + K^0 E_{01} = K^3 E_{32} + K^0 E_{02} = 0 \\ K^3 E_{33} + K^0 E_{03} = -(K^3 E_{30} + K^0 E_{00}) = \frac{1}{2} K^3 (E_{11} + E_{22} + E_{33} - E_{00}) \end{cases}$$

$$K^3 = K^0 = K \Rightarrow \begin{cases} E_{31} + E_{01} = E_{32} + E_{02} = 0 \\ E_{33} + E_{03} = -(E_{30} + E_{00}) = \frac{1}{2} (E_{11} + E_{22} + E_{33} - E_{00}) \end{cases}$$

$$\Rightarrow \begin{cases} E_{01} = -E_{31}, & E_{02} = -E_{32} \\ E_{22} = -E_{11}, & E_{03} = -\frac{1}{2} (E_{33} + E_{00}) \end{cases}$$

$E_{01}, E_{02}, E_{22}, E_{03}$
dependent on other
polarizations.

Residual gauge freedom: $\left. \begin{aligned} \epsilon_{13} &\rightarrow \epsilon_{13} + k \epsilon_1 \\ \epsilon_{23} &\rightarrow \epsilon_{23} + k \epsilon_2 \\ \epsilon_{33} &\rightarrow \epsilon_{33} + 2k \epsilon_3 \\ \epsilon_{00} &\rightarrow \epsilon_{00} - 2k \epsilon_0 \end{aligned} \right\} \begin{array}{l} \text{Can choose} \\ \boxed{\epsilon_{13} = \epsilon_{23} = \epsilon_{33} = \epsilon_{00} = 0} \\ \rightarrow \text{unphysical} \\ \text{polarizations.} \end{array}$

\Rightarrow Only two components ($\epsilon_{11}, \epsilon_{12}$) have independent physical significance.

$$\left. \begin{aligned} \epsilon_{01} = -\epsilon_{31} = -\epsilon_{13} = 0 \\ \epsilon_{02} = -\epsilon_{32} = -\epsilon_{23} = 0 \\ \epsilon_{03} = -\frac{1}{2}(\epsilon_{33} + \epsilon_{00}) = 0 \end{aligned} \right\} \begin{array}{l} \text{from harmonic conditions, and} \\ \text{above gauge choice.} \end{array}$$

The polarization tensor in this gauge takes the form

$$\epsilon_{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \epsilon_+ & \epsilon_x & 0 \\ 0 & \epsilon_x & -\epsilon_+ & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}_{\mu\nu} \quad \text{for some } \epsilon_+, \epsilon_x$$

Notice that in this gauge, $k^\mu \epsilon_{\mu\nu} = 0 \leftarrow \text{transverse}$
and $\epsilon^\mu{}_\mu = 0 \leftarrow \text{traceless}$

This is called transverse, traceless gauge.

Caution:

Notice that we used the equations of motion with $T_{\mu\nu} = 0$ in order to deduce $k_\mu k^\mu = 0$, which led to $k^3 = k^0$ here. If $T_{\mu\nu} \neq 0$, we might not be able to simultaneously satisfy the equations of motion (i.e. the linearized Einstein eqs.) and the transverse + traceless conditions on $h_{\mu\nu}$.

Helicity of Gravitational Waves

Consider a rotation by angle θ about the x^3 -axis,

$$\Lambda^\mu{}_\nu = \begin{pmatrix} 1 & & & \\ & \cos\theta & \sin\theta & \\ & -\sin\theta & \cos\theta & \\ & & & 1 \end{pmatrix}, \quad (\Lambda^{-1})^\mu{}_\nu = \begin{pmatrix} 1 & & & \\ & \cos\theta & -\sin\theta & \\ & \sin\theta & \cos\theta & \\ & & & 1 \end{pmatrix}$$

The polarization tensor transforms as

$$E_{\mu\nu} \rightarrow E'_{\mu\nu} = (\Lambda^{-1})^\alpha{}_\mu (\Lambda^{-1})^\beta{}_\nu E_{\alpha\beta}$$

Defining $\begin{cases} E_\pm \equiv E_{11} \pm iE_{12} = -E_{22} \pm iE_{21} \\ f_\pm \equiv E_{31} \pm iE_{32} = -E_{01} \mp iE_{02} \end{cases}$,

it is straightforward to check that under the rotation,

$$E'_\pm = \exp(\pm 2i\theta) E_\pm \quad \leftarrow \text{helicity } \pm 2$$

$$f'_\pm = \exp(\pm i\theta) f_\pm \quad \leftarrow \text{helicity } \pm 1$$

$$E'_{33} = E_{33}, \quad E'_{00} = E_{00} \quad \leftarrow \text{helicity } 0$$

Any plane wave which transforms as $\psi' = e^{ih\theta} \psi$ under a rotation by θ about the direction of motion is said to have helicity h .

In our analysis of plane wave solutions for h_{μν}, we chose $k^1 = k^2 = 0$, so motion is in the x^3 direction.

We also found that the physical components of $E_{\mu\nu}$ were E_{11} and E_{12} , which could be replaced by the linear combinations E_\pm .

Hence, gravitational waves are decomposed into helicity $\pm 2, \pm 1, 0$ parts, but only the helicity ± 2 parts are physical.

Motion of particles in a gravitational wave

zee 1x14

Consider a particle initially at rest, $\frac{dx^0}{d\tau} = 1$, $\frac{d\vec{x}}{d\tau} = 0$.

The particle follows the geodesic equation,

$$\frac{d^2 x^\mu}{d\tau^2} + \Gamma_{\nu\lambda}^\mu \frac{dx^\nu}{d\tau} \frac{dx^\lambda}{d\tau} = 0$$

$$\rightarrow \frac{d^2 x^\mu}{d\tau^2} + \Gamma_{00}^\mu \approx 0 \text{ near the initial instant}$$

$$\Gamma_{00}^\mu = \frac{1}{2} \eta^{\mu\lambda} (\partial_0 h_{0\lambda} + \partial_0 h_{\lambda 0} - \partial_\lambda h_{00}) + \mathcal{O}(h^2)$$

For a plane wave in transverse-traceless gauge, $\epsilon_{0\lambda} = \epsilon_{\lambda 0} = 0$.

$$\rightarrow \Gamma_{00}^\mu \approx 0.$$

$$\frac{d^2 x^\mu}{d\tau^2} \approx 0, \leftarrow \text{particle at rest remains at rest, at least for short times.}$$

If a particle does not respond to a passing gravitational wave, then how would such a wave be detected?

Answer: Consider a collection of particles.

A physical gravitational wave would be in the form of a wavepacket, i.e. a superposition of plane waves.

For example, consider a superposition of plane waves in the x^3 -direction

$$k^\mu \sim (k, 0, 0, k)$$

$$h_{\mu\nu} = \int dk \tilde{f}(k) e^{ik(z-t)} \epsilon_{\mu\nu}(k) + \text{c.c.}$$

\leftarrow complex conjugate

Suppose $\epsilon_{\mu\nu}(k) = \epsilon_{\mu\nu}$ independent of k .

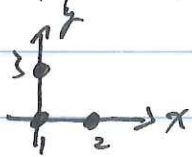
Then $h_{\mu\nu}$ has the form $h_{\mu\nu} \equiv f(z-t) \epsilon_{\mu\nu} + \text{c.c.}$



In transverse-traceless gauge,

$$\epsilon_{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \epsilon_t & \epsilon_x & 0 \\ 0 & \epsilon_x & -\epsilon_t & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Suppose there are three particles in the x - y plane:



$$x_1 = (0, 0, 0)$$

$$x_2 = (a, 0, 0)$$

$$x_3 = (0, 0, a)$$

Assume a is small

compared to the width of the wavepacket.

The proper distance between these points is:

$$(\Delta S_{12})^2 = g_{xx} a^2 = (1 + h_{xx}) a^2$$

$$|\Delta S_{12}| = a \sqrt{1 + h_{xx}} \approx a \left(1 + \frac{1}{2} (f(z-t) \epsilon_t + c.c.) \right)$$

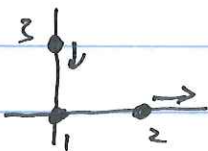
$$= a \left(1 + \text{Re}(f(z-t) \epsilon_t) \right)$$

$$(\Delta S_{13})^2 = g_{zz} a^2 = (1 + h_{zz}) a^2$$

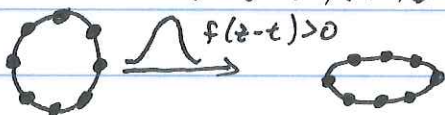
$$|\Delta S_{13}| \approx a \left(1 - \frac{1}{2} (f(z-t) \epsilon_t + c.c.) \right)$$

$$= a \left(1 - \text{Re}(f(z-t) \epsilon_t) \right)$$

As the distance between 1 and 3 shrinks, the distance between 1 and 2 grows, and vice versa.

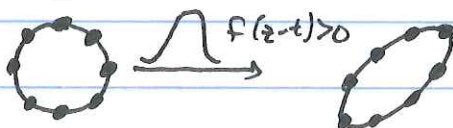


A circular distribution of particles would be distorted into an ellipsoidal shape:



$$\epsilon_t < 0, \epsilon_x = 0$$

This distortion is the basis of gravitational wave searches.



$$\epsilon_t = 0, \epsilon_x < 0$$