More About Black Holes

Wald 12.3
Carroll Ch. 6

Most general known static black hole in 3+1 dim's:

$$ds^2 = -\left(\frac{\Delta - a^2 \sin^2\theta}{\Sigma}\right) dt^2 - \left(\frac{2a \sin^2\theta\,(r^2+a^2-\Delta)}{\Sigma}\right) dt\, d\phi$$
$$+ \left[\frac{(r^2+a^2)^2 - \Delta a^2 \sin^2\theta}{\Sigma}\right] \sin^2\theta \; d\phi^2$$
$$+ \frac{\Sigma}{\Delta} dr^2 + \Sigma\, d\theta^2 ,$$

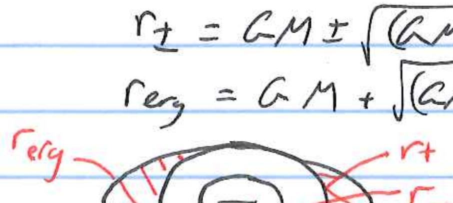
where $\Sigma = r^2 + a^2 \cos^2\theta$
$\Delta = r^2 + a^2 + e^2 - 2Mr$

$a \sim$ angular momentum/mass, $e \sim$ electric charge
$\rightarrow F_{\mu\nu} \neq 0$, but we're not writing it here.

$e = a = 0 \rightarrow$ Schwarzschild (1916)
$a = 0,\ e \neq 0 \rightsquigarrow$ Reissner-Nordstrom (1916, 1918)
$e = 0,\ a \neq 0 \rightsquigarrow$ Kerr (1963)

The Kerr and Reissner-Nordstrom black holes have a more intricate causal structure than the Schwarzschild black hole, e.g. multiple horizons, ring singularity (Kerr).

<u>Kerr</u>: $r = 0,\ \theta = \pi/2$ singularity $\sim x^2 + y^2 = a^2$
$r_\pm = GM \pm \sqrt{(GM)^2 - a^2}$ horizon
$r_{erg} = GM + \sqrt{(GM)^2 - a^2 \cos^2\theta}$ ergosurface



observers cannot be static in ergosphere.

$r = 0$ - timelike (can be avoided)

Black Hole Thermodynamics

In the 1970s, Bekenstein, Carter and Hawking uncovered analogies between black holes and thermodynamic systems. This program included Hawking's discovery of black hole radiation.

0th Law of Thermodynamics: Equilibrium ~ there exists a quantity (temperature, T) that is uniform across the system.

~ surface gravity ($\kappa = \frac{1}{4GM}$ for Schwarzschild B.H.) is uniform on the horizon.

On dimensional grounds, we expect $T = \frac{const.}{GM} \propto \kappa$.
(for Schwarzschild)

1st Law: $dE = TdS + dW_{(ork)}$

$\sim \; dM = TdS + \cancel{dW}$ (assume no work on the BH)

$= \frac{const}{GM} dS$ (for Schwarzschild BH)

Integrate $\rightarrow S = \frac{GM^2}{2 \times const.} = \frac{R_S^{\;2}}{8(const) \times G}$, where $R_S = 2GM$

Horizon area $A_H = 4\pi R_S^2$

$$S = \frac{A_H}{32\pi \cdot const \cdot G} \propto \frac{A_H}{G}$$

Entropy varies like an area, not extensively like a volume! It is as if a dimension is lost when counting independent microscopic configurations w/ black holes $\rightarrow$ "Holography"

2nd Law: $\frac{\Delta S}{\Delta t} \geq 0 \;\sim\; \frac{\Delta A_H}{\Delta t} \geq 0$ Black holes grow (classically)

To determine the temperature of a black hole, we consider statistical mechanics, or statistical field theory, in the black hole background.

Partition function $Z = \mathrm{Tr}\, e^{-\beta H} = \sum_{|\psi_n\rangle} \langle \psi_n | e^{-\beta H} | \psi_n \rangle$

$$|\psi_n\rangle = \text{eigenstates of } H, \quad \beta = \frac{1}{k_B T}$$

In quantum mechanics, $|\psi(t)\rangle = e^{-iHt/\hbar} |\psi(0)\rangle$

$$\langle \psi_n(0) | \psi_n(t) \rangle = \langle \psi_n(0) | e^{-iHt/\hbar} | \psi_n(0) \rangle$$

= amplitude for state to return to itself after time t.

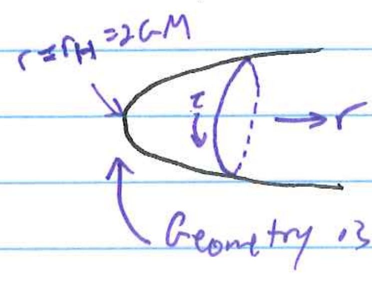
Comparing with the partition function, it is as if $\tau = it/\hbar$, and the sum over states includes states periodic in $\tau \to \tau + \beta$.

→ Do quantum mechanics / field theory in a "Euclidean" black hole background with $t \to -i\hbar\tau$, on a Euclidean time circle with $\tau \sim \tau + \beta$ identified.

$(\hbar = 1)$ $\quad ds^2 = f(r)\, d\tau^2 + f(r)^{-1} dr^2 + r^2 (d\theta^2 + \sin^2\theta\, d\phi^2)$,

where $f(r) = 1 - \frac{2GM}{r}$ (Schwarzschild)

Consider only the (r, t) projection of the geometry



Consider the circle $\tau \in [0, \beta)$ at $r = r_H + \epsilon$ near the horizon, i.e. $\epsilon \ll r_H$.

Proper circumference of circle: $\beta_* = \sqrt{f(r)}\,\beta$

- from $d\tau^2$ part of metric.

$$\beta_* = \sqrt{f(r_H+\epsilon)}\,\beta \approx \sqrt{\cancel{f(r_H)} + f'(r_H)\,\epsilon}\,\beta$$

$$= \sqrt{f'(r_H)}\,\sqrt{\epsilon}\,\beta.$$

If the geometry is nonsingular at the horizon, we can also calculate the proper circumference as $\beta_* = 2\pi \Delta r_*$, where Δr_* is the proper radius,

$$\Delta r_* \approx \int_{r_H}^{r_H+\epsilon} \frac{dr}{\sqrt{\cancel{f(r_H)} + f'(r_H)(r - r_H)}}$$

$$= \frac{2\sqrt{\epsilon}}{\sqrt{f'(r_H)}}$$

Equating the two evaluations of β_*, we have

$$\beta_* = \sqrt{f'(r_H)}\,\sqrt{\epsilon}\,\beta = \frac{2\sqrt{\epsilon}}{\sqrt{f'(r_H)}} \cdot 2\pi$$

$$\Rightarrow \frac{1}{k_B T} = \beta = \frac{4\pi}{f'(r_H)} = \frac{4\pi}{(2GM/r_H^2)} = 8\pi G M$$

$\rightarrow$ Temperature of Hawking radiation. Including c, $\hbar$!

Hawking $\rightarrow$ $$\boxed{T_H = \frac{\hbar c^3}{8\pi G M k_B}}$$

Unruh Radiation

A static observer ($r=\text{const.}$) near the black hole horizon observes a blue-shifted temperature

$$\beta_* = \frac{4\pi}{\sqrt{f'(r_H)}}\sqrt{\epsilon}$$

$$= 4\pi\sqrt{\frac{(2GM)^2}{2GM}}\sqrt{(r-2GM)} = 4\pi\sqrt{2GM(r-2GM)}$$

Define $r = 2GM + \frac{\rho^2}{8GM}$, $\rho=0$ horizon

$$ds^2 = -\frac{\rho^2}{16GM}dt^2 + d\rho^2 + r(\rho)^2(d\theta^2+\sin^2\theta\, d\phi^2)$$

$$\equiv -\rho^2 d\tilde{t}^2 + d\rho^2 + r(\rho)^2(d\theta^2+\sin^2\theta\, d\phi^2)$$

where $d\tilde{t} = \frac{dt}{\sqrt{16GM}}$

The near-horizon geometry of the black hole is Rindler space, i.e. Minkowski space in an accelerated coordinate system.

proper acceleration of observer at $\rho>0$, $a = 1/\rho$.

In these coordinates, $\beta_* = 4\pi\sqrt{2GM\left(\cancel{2GM} + \frac{\rho^2}{8GM} - \cancel{2GM}\right)}$

$$= 2\pi\rho = \frac{2\pi}{a}$$

We conclude that an accelerated observer in the Minkowski space vacuum observes a thermal background with $\boxed{T_* = \frac{\hbar}{ck_B}\cdot\frac{a}{2\pi}}$, a = proper acceleration.