Time-Dependent Spherically Symmetric Fields

Weinberg 11.7

The most general spherically symmetric metric can be written

$$ds^2 = -C(r,t)\,dt^2 + D(r,t)\,dr^2 + 2E(r,t)\,dr\,dt + F(r,t)\,r^2\left(d\theta^2 + \sin^2\theta\, d\varphi^2\right)$$

We can set $F(r,t)=1$ by rescaling $r \to rF^{1/2}$.
We can set $E(r,t)=0$ by redefining the time coordinate:

$$dt \to \eta(r,t)\left[C(r,t)\,dt - E(r,t)\,dr\right]$$

where η is chosen so that dt is a total differential, i.e.

$$\frac{\partial}{\partial r}\left[\eta(r,t)\,C(r,t)\right] = -\frac{\partial}{\partial t}\left[\eta(r,t)\,E(r,t)\right]$$

Then,

$$ds^2 \to \underbrace{\eta^{-2}C^{-1}}_{B(r,t)}dt^2 + \underbrace{\left(D + C^{-1}E^2\right)}_{A(r,t)}dr^2 + r^2\left(d\theta^2 + \sin^2\theta\, d\varphi^2\right)$$

$$\Rightarrow \boxed{ds^2 = -B(r,t)\,dt^2 + A(r,t)\,dr^2 + r^2\left(d\theta^2 + \sin^2\theta\, d\varphi^2\right)}$$

Exercise: The nonvanishing components of the Ricci tensor are

$$R_{rr} = \frac{B''}{2B} - \frac{B'^2}{4B^2} - \frac{A'B'}{4AB} - \frac{A'}{rA} - \frac{\ddot{A}}{2B} + \frac{\dot{A}\dot{B}}{4B^2} + \frac{\dot{A}^2}{4AB}$$

$$R_{tt} = -\frac{B''}{2A} + \frac{B'A'}{4A^2} - \frac{B'}{rA} + \frac{B'^2}{4AB} + \frac{\ddot{A}}{2A} - \frac{\dot{A}^2}{4A^2} - \frac{\dot{B}\dot{A}}{4AB}$$

$$R_{\theta\theta} = -1 + \frac{1}{A} - \frac{rA'}{2A^2} + \frac{rB'}{2AB} \qquad \Big\| \quad R_{tr} = R_{rt} = -\frac{\dot{A}}{rA} \quad \Big\| \quad R_{\varphi\varphi} = \sin^2\theta\, R_{\theta\theta}$$

where $A' = \frac{\partial}{\partial r}A(r,t)$, $\dot{A} = \frac{\partial}{\partial t}A(r,t)$, etc.

In empty space, $R_{\mu\nu}=0$.

$$R_{tr}=0 \Rightarrow \boxed{\dot{A}=0} \Rightarrow \ddot{A}=0$$

Then all time derivatives disappear from the Einstein eqns.

$$R_{rr}\cdot\frac{B}{A}+R_{tt} = -\frac{(A'B+B'A)}{rA^2} = 0$$

$$\Rightarrow \boxed{(AB)'=0}$$

$$R_{\theta\theta}=0 = -1+\left(\frac{r}{A}\right)'+\cancelto{0}{\frac{r(AB)'}{2A^2B}}$$

$$\Rightarrow \boxed{\left(\frac{r}{A}\right)'=1}$$

Solution: $A=\left(1-\frac{2GM}{r}\right)^{-1}$, $B=f(t)\left(1-\frac{2GM}{r}\right)$

We can set $f(t)=1$ by redefining the time coordinate

$$t \to \int^t f^{1/2}(t)\,dt$$

$$\Rightarrow \boxed{ds^2=-\left(1-\frac{2GM}{r}\right)dt^2+\left(1-\frac{2GM}{r}\right)^{-1}dr^2+r^2(d\theta^2+\sin^2\theta\, d\phi^2)}$$

We have proved <u>Birkhoff's Theorem</u>: a spherically symmetric gravitational field in empty space must be static, with metric given by the Schwarzschild solution.

A consequence of Birkhoff's Theorem is that there is no spherically symmetric gravitational radiation.

Another consequence is that inside a spherically symmetric shell, spacetime = Minkowski.

Cosmology

Weinberg 14, 15

Study of the large-scale evolution of the universe

Cosmological Principle: All positions in the universe are essentially equivalent
(averaging over regions of size ~10^9 light years today)
→ The universe is homogeneous and isotropic.

Homogeneity – roughly, translation invariance
More precisely: At each point X there exist isometries that take X to any point in a neighborhood of X.

Isotropy – roughly, rotation invariance
More precisely: At each point X there exist isometries that leave X fixed but such that the derivatives of Killing vectors $\xi_{\lambda;\nu} = -\xi_{\nu;\lambda}$ can take any values.

Isotropy about every point implies homogeneity. (Weinberg 13.1)

Friedmann-Robertson-Walker Metric (FRW):
foliate spacetime into hypersurfaces of constant time which are homogeneous and isotropic.

$$ds^2 = -dt^2 + \underbrace{R^2(t)}_{\text{time-dependent scale factor}} \underbrace{h_{ij}(\vec{x})}_{\text{3-space metric}} dx^i dx^j$$

Note: In differential geometry, coordinates such that the metric takes the form

$$ds^2 = -dt^2 + D(r,t)\,dr^2 + G(r,t)\left(d\theta^2 + \sin^2\theta\, d\varphi^2\right)$$

are called <u>Gaussian normal coordinates</u>.

The FRW metric is an example. Isotropy → we can find coordinates r, θ, φ about any point such that $h_{ij}\,dx^i dx^j = A(r)\,dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\varphi^2)$

$$\rightsquigarrow ds^2 = -dt^2 + R^2(t)\underbrace{\left[A(r)\,dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\varphi^2)\right]}_{h_{ij}\,dx^i dx^j}$$

Consider the 3-dimensional geometry described by h_{ij}.

Ricci tensor:

$$R^{(3)}_{rr} = -\frac{1}{rA}\frac{dA}{dr}, \quad R^{(3)}_{\theta\theta} = -1 + \frac{1}{A} - \frac{r}{2A^2}\frac{dA}{dr}$$

$$R^{(3)}_{\varphi\varphi} = \sin^2\theta\, R^{(3)}_{\theta\theta}$$

Curvature scalar:

$$R^{(3)} = h^{ij}R^{(3)}_{ij} = \frac{1}{A}\left(-\frac{1}{rA}\frac{dA}{dr}\right) + \frac{1}{r^2}\left(-1 + \frac{1}{A} - \frac{r}{2A^2}\frac{dA}{dr}\right)$$

$$+ \frac{1}{r^2\sin^2\theta}\cdot\sin^2\theta\left(-1 + \frac{1}{A} - \frac{r}{2A^2}\frac{dA}{dr}\right)$$

$$= \frac{2}{r^2}\left(-1 + \frac{d}{dr}\left(\frac{r}{A}\right)\right)$$

Homogeneity → $R^{(3)}$ is independent of r → $R^{(3)} \equiv -6K$

$$-3kr^2 = -1 + \frac{d}{dr}\left(\frac{r}{A}\right)$$

Integrate!

$$\frac{r}{A} = r - kr^3 + \text{const.}$$

$$A = \frac{1}{1-kr^2+\frac{\text{const}}{r}} \qquad \text{const.}=0 \text{ to avoid singularity}$$

$$\Longrightarrow \boxed{A = \frac{1}{1-kr^2}}$$

We have now determined the form of the FRW metric!

$$\boxed{ds^2 = -dt^2 + R^2(t)\left[\frac{dr^2}{1-kr^2} + r^2\left(d\theta^2 + \sin^2\theta\, d\varphi^2\right)\right]}$$

By rescaling $r \to \alpha r$, $R \to \frac{1}{\alpha} R$, we can choose $\boxed{k = +1, -1, \text{ or } 0}$.

$$k=0 \longrightarrow ds^2 = -dt^2 + R^2(t)\left[dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\varphi^2)\right]$$

$$= -dt^2 + R^2(t)\left[dx^2 + dy^2 + dz^2\right]$$

Constant-t slices of the $k=0$ FRW spacetime are flat $\rightsquigarrow$ The $k=0$ spacetime is called a flat FRW universe.

$$k=1 \rightarrow \quad ds^2 = -dt^2 + R^2(t)\underbrace{\left[\frac{dr^2}{1-r^2} + r^2\left(d\theta^2 + \sin^2\theta\, d\varphi^2\right)\right]}_{\text{3-sphere of radius } 1.}$$

3-sphere: $x^2+y^2+z^2+w^2=1$

$$ds^2 = dx^2+dy^2+dz^2+dw^2$$

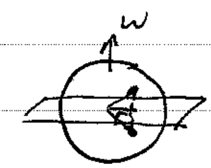
$(x,y,z) \rightarrow (r,\theta,\varphi)$

$$r^2+w^2=1$$

$$r\,dr + w\,dw = 0 \rightarrow dw^2 = \frac{r^2dr^2}{w^2} = \frac{r^2dr^2}{1-r^2}$$

$$\rightarrow \quad ds_{(3)}^2 = \frac{dr^2}{1-r^2} + r^2\left(d\theta^2+\sin^2\theta\, d\varphi^2\right) \quad \checkmark$$

Each r corresponds to two points $w = \pm\sqrt{1-r^2}$



$0 \le r \le 1$ covers half of the 3-sphere $\times 2$

$r = \sin\psi$, $dr = \cos\psi\, d\psi$, $1-r^2 = \cos^2\psi$

$$R^2 ds_{(3)}^2 = R^2\left[d\psi^2 + \sin^2\psi\left(d\theta^2+\sin^2\theta\, d\varphi^2\right)\right]$$

$$x = R\sin\psi\sin\theta\cos\varphi$$
$$y = R\sin\psi\sin\theta\sin\varphi$$
$$z = R\sin\psi\cos\theta$$
$$w = R\cos\psi \qquad \leftarrow \quad 0 \le \psi \le \pi \text{ covers whole sphere}$$

$$\sqrt{g} = R^3\sin^2\psi\sin\theta$$

Proper volume of space w/ metric $R^2 ds_{(3)}^2$:

$$V = \int_0^{2\pi} d\varphi \int_0^{\pi} d\theta \int_0^{\pi} d\psi\; R^3\sin^2\psi\sin\theta = 2\pi R^3\cdot\frac{\pi}{2}\cdot 2 = \boxed{2\pi^2R^3}$$

$\rightarrow$ space is <u>closed</u> (finite volume)

$k=-1$: $ds_{(3)}^2 = \frac{dr^2}{1+r^2} + r^2(d\theta^2 + \sin^2\theta\, d\varphi^2)$

3D hyperboloid

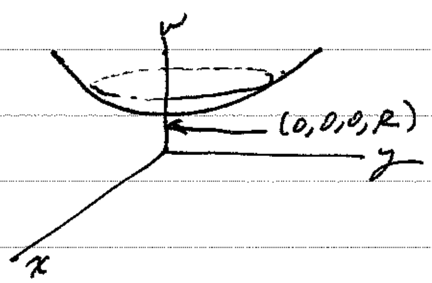
$w^2 - x^2 - y^2 - z^2 = 1 \qquad = w^2 - r^2$

$ds^2 = -dw^2 + dx^2 + dy^2 + dz^2$

$w\,dw - r\,dr = 0 \rightarrow dw^2 = \frac{r^2 dr^2}{w^2} = \frac{r^2 dr^2}{1+r^2}$

$\rightarrow R^2 ds_{(3)}^2 = R^2\left[\frac{dr^2}{1+r^2} + r^2(d\theta^2 + \sin^2\theta\, d\varphi^2)\right]$ ✓

$z=0$ slice: $w^2 - x^2 - y^2 = R^2$



hyperboloid of revolution

Transformation $(0,0,0,R) \rightarrow (x,y,z,w)$ Lorentz transformation

Metric independent of location on hyperboloid

$\rightarrow$ homogeneous + isotropic about every pt.

Comoving Coordinates

Consider the worldline $x^\mu = (t, r = \text{const}, \varphi = \text{const}, \theta = \text{const})$.
We will show that these are geodesics.

$$\frac{dx^\mu}{d\tau} = (1,0,0,0) \qquad ds^2 = -d\tau^2 = -dt^2$$

Geodesic Eqn: $$\frac{d^2x^\mu}{d\tau^2} + \Gamma^\mu_{\alpha\beta}\frac{dx^\alpha}{d\tau}\frac{dx^\beta}{d\tau} = 0$$

$$\Gamma^\mu_{tt} = \frac{1}{2} g^{\mu\lambda}\left(\frac{\partial g_{\lambda t}}{\partial t} + \frac{\partial g_{t\lambda}}{\partial t} - \frac{\partial g_{tt}}{\partial x^\lambda}\right) = 0$$

$\rightarrow x^0 = t,\ r = \text{const},\ \theta = \text{const},\ \varphi = \text{const}$ is a geodesic.
These geodesics are lines of constant spatial coordinates, with proper time along the geodesics given by the coordinate t.

Such coordinates are called <u>comoving coordinates.</u>

Sit at $r = 0$, consider another object at $r = r_0$, $\theta = \varphi = 0$.
proper distance: $D = R(t)\int_0^{r_0}\frac{dr}{\sqrt{1-Kr^2}} = R(t)\cdot\text{const.}$

$$\frac{dD}{dt} = \text{const.} \times \dot{R} \longrightarrow \frac{dD}{dt} = \frac{D}{R}\cdot\dot{R} \equiv D(t)\,H(t)$$

$$\boxed{H(t) \equiv \frac{\dot{R}(t)}{R(t)}}\ ,\quad \boxed{\frac{dD}{dt} = HD}$$

Hubble Law

Cosmological
Redshift

Suppose we sit at $r=0$ and receive a light signal from r_e, $\theta_e = \varphi_e = 0$.

$$0 = -dt^2 + R^2(t)\frac{dr^2}{1-kr^2} \quad \text{null trajectory}$$

We measure time by coordinate t.

$$\text{Null trajectory:} \int_{t_{emit}}^{t_{receive}} \frac{dt}{R(t)} = \int_0^{r_e} \frac{dr}{\sqrt{1-kr^2}} = \begin{cases} \sin^{-1} r_e, & k=+1 \\ r_e, & k=0 \\ \sinh^{-1} r_e, & k=-1 \end{cases}$$

Another signal sent Δt_{emit} later, received at $t_{rec} + \Delta t_{rec}$.

$$\int_{t_e+\Delta t_e}^{t_r+\Delta t_r} \frac{dt}{R(t)} = \int_0^{r_e} \frac{dr}{\sqrt{1-kr^2}} = \int_{t_e}^{t_r} \frac{dt}{R(t)}$$

$$\rightarrow \frac{\Delta t_r}{R(t_r)} = \frac{\Delta t_e}{R(t_e)}$$

Frequency $$\boxed{\frac{\nu_{received}}{\nu_{emitted}} = \frac{R(t_{emitted})}{R(t_{received})}}$$

Redshift Factor

$$\boxed{z \equiv \frac{\lambda_{rec} - \lambda_{emit}}{\lambda_{emit}} = \frac{R(t_{rec})}{R(t_{emit})} - 1}$$