Power Radiated in Gravitational Waves

Consider a plane wave $h_{\mu\nu} = e^{i k \cdot x} + e^{-i k \cdot x}$. The energy-momentum "tensor" in the wave pulse is given by $\tau_{\mu\nu} = \frac{1}{8 \pi G} \left[ R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R \right]$. The expansion of $\tau_{\mu\nu}$ in $E_{\mu\nu}$ is messy, but simplifies if we consider the spatial average over distances $\gg \frac{1}{k^2}$. $\langle \tau_{\mu\nu} \rangle$. Then we can use $\langle e^{i k \cdot x} \rangle = 0$, which eliminates a number of terms.

$\langle \tau_{\mu\nu} \rangle \approx \text{spatial average over pulse.}$

Using $R_{\mu\nu} = \frac{1}{2} \left( \partial_{\mu} \partial_{\nu} h^{\alpha\beta} - \partial_{\alpha} h_{\nu}^{\beta} - \partial_{\beta} h_{\mu}^{\alpha} + \partial_{\alpha} h_{\beta}^{\mu} \right)$,

$R^{\mu\nu} = \partial_{\nu} h^{\mu} - \partial_{\mu} h^{\nu}

R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R^{\mu\nu} = \frac{1}{2} \left( \partial_{\nu} \partial_{\mu} h^{\alpha\beta} - \partial_{\alpha} h_{\mu}^{\beta} - \partial_{\beta} h_{\nu}^{\alpha} + \partial_{\alpha} h_{\beta}^{\mu} \right)
- \frac{1}{2} \left( \partial_{\nu} \partial_{\mu} h^{\rho} + \partial_{\mu} \partial_{\nu} h^{\rho} \right)$

Comparing with our earlier analysis of plane waves, $R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R^{\mu\nu} = 0$, which confirms the linearized approximation of the vacuum Einstein eqns.
Using our earlier expressions for \( R_{\mu\nu}^{(4)} \) and \( \langle e^{\pm 2i\lambda x} \rangle = 0 \), we obtain (Exercise):

\[
\langle R_{\mu\nu}^{(4)} \rangle = \frac{k_m k_u}{2} (e^{\lambda^2} \delta_{\mu\nu} - \frac{1}{2} |e_{\lambda}^x|^2)
\]

Using \( \lambda^2 = 0 \), \( \langle R^{(4)} \rangle = 0 \)

Hence,

\[
\langle t_{\mu\nu} \rangle = \frac{k_m k_u}{16\pi G_n} (e^{\lambda^2} \delta_{\mu\nu} - \frac{1}{2} |e_{\lambda}^x|^2)
\]

We have previously found the gravitational radiation field far from a locally oscillating source. Summarizing our earlier results:

\[
T_{\mu\nu}(x,t) = e^{-i\omega t} \tilde{T}_{\mu\nu}(x', \omega) + c.c.
\]

\[
\tilde{T}_{\mu\nu}(x', \omega) = h_{\mu\nu} - \frac{1}{2} \gamma_{\mu\nu} h^p
\]

\[
= -\frac{\lambda}{4\pi r} \tilde{T}_{\mu\nu}(x', \omega) e^{i\omega t} e^{i\mathbf{k} \cdot \mathbf{x}}
\]

where we had, in our new language, \( R_{\mu\nu}^{(4)} - \frac{1}{2} g_{\mu\nu} R^{(4)} = \frac{\lambda}{2} T_{\mu\nu} \)

\[
\Rightarrow \lambda = -16\pi G_n
\]

\[
\Rightarrow \tilde{T}_{\mu\nu}(x', \omega) = \frac{4\pi}{\lambda} e^{-i\omega(t-r)} \tilde{T}_{\mu\nu}(x', \omega) \quad a + \mathbf{k} = \mathbf{n} \hat{\mathbf{x}}
\]

\[
\tilde{S}_{\mu\nu}(x', \omega) = \frac{4\pi}{\lambda} (\tilde{T}_{\mu\nu}(x', \omega) - \frac{1}{2} \gamma_{\mu\nu} \tilde{T}(x', \omega))
\]
\[ E_{\mu} + E_{\mu} = \frac{16 \alpha^2}{r^2} \left( \hat{T}_{\mu} + \hat{T'}_{\mu} - \frac{1}{2} \hat{T''}_{\mu} - \frac{2}{\alpha^2} \frac{\partial}{\partial t} + \frac{2}{\alpha^2} \frac{\partial}{\partial y} \right) \]

\[ = \frac{16 \alpha^2}{r^2} \hat{T}_{\mu} + \hat{T'}_{\mu} \]

\[ |E_{\mu}|^2 = \frac{16 \alpha^2}{r^2} \left( \hat{T}_{\mu} + \hat{T'}_{\mu} - \frac{1}{2} \delta_{\mu} \delta_{\mu} \right)^2 = \frac{16 \alpha^2}{r^2} |\hat{T}_{\mu}|^2 \]

\[ \langle T_{\mu\nu} \rangle = \frac{k_{\mu, \nu}}{16 \pi \alpha^2} \cdot \frac{16 \alpha^2}{r^2} \left( \hat{T}_{\mu} + \hat{T'}_{\mu} - \frac{1}{2} \Delta_{\mu} |\hat{T}_{\mu}|^2 \right) \]

Away from source, \(\partial t^{\mu\nu} = 0\)

\[ \partial t^{00} = - \partial_t t' \]

\[ \int E: \hat{E} \times t' \]

\[ P = - \frac{dE}{dt} = \int \hat{E} \times \hat{t}' \]

\[ P = \text{radiated power in gravitational radiation from vol. V} \]

\[ \frac{dP}{dS} = r^2 \hat{r} \hat{t}' \langle t'0 \rangle \]

\[ = \frac{r^2 (E \cdot \hat{r}) E^0}{16 \pi \alpha^2} \left( \hat{T}_{\mu} + \hat{T'}_{\mu} - \frac{1}{2} |\hat{T}_{\mu}|^2 \right) \]

\[ = \frac{G \omega^2}{\alpha} \left( \hat{T}_{\mu} + \hat{T'}_{\mu} - \frac{1}{2} |\hat{T}_{\mu}|^2 \right) \]
Recall from earlier that with the relevant approximations
\[ \tilde{T}_{ij}(\hat{x},\hat{w}) = -\frac{\omega^2}{2} D_{ij}(w) \]
where \( D_{ij} \) is the quadrupole moment
\[ D_{ij}(w) = \int d^2 x \ x^i x^j \hat{T}_{00}(x, w) \]

Using \( \hat{D}_\mu T^{\mu\nu} = 0 \) (to lowest order),
\[ K_\mu \tilde{T}^{\mu\nu}(x, w) = 0 \]
\[ \Rightarrow \ K_0 \tilde{T}^{0i} + K_i \tilde{T}^{ji} = 0 \]
\[ \tilde{T}^{0i} = -\frac{\tilde{\omega}_i}{K_0} \tilde{T}^{ji} = \frac{\tilde{\omega}_i}{K_0} \tilde{T}^{ji} \]
\[ K_0 \tilde{T}^{00} + \tilde{\omega}_i \tilde{T}^{0i} = 0 \]
\[ \tilde{T}^{00} = -\frac{\tilde{\omega}_i \tilde{T}^{0i}}{K_0} = \frac{\tilde{\omega}_i \tilde{T}^{0i}}{K_0} \]

Hence, we can express all components of \( \tilde{T}^{\mu\nu} \) in terms of \( \tilde{T}^{ij} \), and hence in terms of \( D_{ij} \).
\[ \tilde{T}^{0i} = -\frac{\omega^2}{2} x^j D^{ij} \]
\[ \tilde{T}^{00} = -\frac{\omega^2}{2} x^i x^j D^{ij} \]

\[ \frac{d\phi}{d\Omega} = \frac{G_4 \omega^2}{\pi} \left( \frac{\omega^2}{2} \right)^2 \left( \frac{\hat{x}_i \hat{x}_j \hat{x}_k \hat{x}_l}{x^i x^j x^k x^l} D^{ijkl} \right)^2 \]
\[ + D^{ij} D_{ij} - \frac{1}{2} \hat{x}_i \hat{x}_j \hat{x}_k \hat{x}_l D^{ijkl} D^{kl} + \frac{1}{2} \hat{x}_k \hat{x}_l D^{kl} D^{ijkl} \]
\[ + \frac{1}{2} \hat{x}_i \hat{x}_j D^{ij} D^{kl} - \frac{1}{2} D^{ijkl} D^{ij} \]
\[
\frac{dp}{dt} = \frac{G \omega^6}{4 \pi} \left( \frac{1}{2} \hat{x}_i \hat{x}_j \hat{x}_k \hat{x}_l \hat{D}^{i j} \hat{D}^{k l} - 2 \hat{x}_j \hat{x}_k \hat{D}^{i j} \hat{D}^{k i} \right.
\]
\[
+ \hat{D}^{i j} \hat{D}^{i j} + \frac{1}{2} \hat{x}_k \hat{x}_l \hat{D}^{i k} \hat{D}^{j l} + \frac{1}{2} \hat{x}_k \hat{x}_l \hat{D}^{i i} \hat{D}^{j k} \hat{D}^{j l}
\]
\[
- \frac{1}{2} \hat{D}^{i i} \hat{D}^{i i} \right)
\]

Total Power radiated: \( P = \int d\Omega \frac{dp}{dt} = \frac{27}{5} \int d\Omega [d(\cos \theta)] \frac{dp}{dt} \)

Use \( \int d\Omega d(\cos \theta) = 4 \pi \)

\( \int d\Omega d(\cos \theta) \hat{x}_i \hat{x}_j = \frac{4 \pi}{5} \delta_{ij} \)

\( \int d\Omega d(\cos \theta) \hat{x}_i \hat{x}_j \hat{x}_k \hat{x}_l = \frac{4 \pi}{55} \left( \delta_{ij} \delta_{kl} + \delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk} \right) \)

\[
\Rightarrow P = \frac{27 \omega^6}{5} \left( D_{i j}(\omega) D_{i j}(\omega) - \frac{1}{2} |D_{i i}(\omega)|^2 \right)
\]

This is the total power radiated at one frequency in the quadrupole approximation W.R.T.
Binary Star System

Assume a circular Newtonian orbit:

\[
\frac{GM^2}{(2R)^2} = \frac{Mv^2}{R}
\]

\[v = \frac{2\pi R}{T} \Rightarrow \frac{GM}{4} \frac{T^2}{\pi^2} = \frac{(2\pi)^2 R^3}{\pi^2} \]

\[T = \text{period of orbit}\]

Let \( T = \frac{2\pi}{T} \) \( \Rightarrow \) \( \frac{GM}{4\pi^2} = R^3 \)

\[\text{Shr. 1: } x_1 = R \cos RT, \quad y_1 = R \sin RT, \quad z_1 = 0\]

\[\text{Shr. 2: } x_2 = -R \cos RT, \quad y_2 = -R \sin RT, \quad z_2 = 0\]

\[
T^\infty = M \left( 5\delta(x) \delta(x - R \cos RT) \delta(y - R \sin RT) \right.
\]
\[+ \delta(z) \delta(z + R \sin RT) \delta(z - R \sin RT) \left. \right) \]

\[D_{x_2} = \int d^3x \, x^2 \, T^\infty(x, t) = 0 = D_{y_2} = D_{z_2}
\]

\[D_{x_1} = \int d^3x \, x^2 \, T^\infty(x, t)
\]
\[= 2MR^2 \cos^2 RT = MR^2 \left( 1 + \frac{1}{4} e^{2iRT} + \frac{1}{4} e^{-2iRT} \right) \]

\[\Rightarrow \quad D_{xx} (2R) = \frac{1}{4} MR^2 \]

\[D_{x_2} = 2MR^2 \sin^2 RT = MR^2 \left( 1 - \frac{1}{2} e^{2iRT} - \frac{1}{2} e^{-2iRT} \right) \]

\[\Rightarrow \quad D_{x_2} (2R) = -\frac{1}{2} MR^2 \]

\[D_{x_3} = \int d^3x \, x_3 \, T^\infty = 2MR^2 \sin RT \cos RT = MR^2 \sin 2RT \]

\[= -iM^2 \left( e^{iRT} - e^{-iRT} \right) \]

\[\Rightarrow \quad D_{x_1} (2R) = -iMR^2 \]
\[ D_{ij} D_{ij} = \left( \frac{MR^2}{2} \right)^2 (1+1+1+1) = M^2 R^4 \]

\[ D_i = D_{xx} + D_{yy} = 0 \]

Power \[ P = \frac{2 \sigma}{5} (2 \pi)^6 M^2 R^4 = \frac{12 \pi}{5} G M^2 R^4 R^6 \]

\[ T^2 = \frac{GM}{4 R^3} \Rightarrow P = \frac{3}{5} \frac{G^4 M^8}{R^8} \]

\[ \text{Power radiated in gravitational radiation} \]
\[ E \propto \tau^{-2/3} \text{ for Newtonian orbits} \]
\[ \frac{1}{T} \frac{dT}{dt} = -\frac{3}{2} \frac{1}{E} \frac{dE}{dt} = \frac{3}{2} \frac{1}{E} P \]

1974 Hulse-Taylor Binary Pulsar
- Supernova remnant \( \text{PSR 1913+16} \)

Change in orbital period measured, mass and size of orbit known

\[ \rightarrow \text{Agrees with prediction of change in period due to energy loss to gravitational radiation!} \]

Note: When comparing with astronomical data, the semimajor axis and other geometric quantities are typically quoted with respect to one of the objects (called the "principal object"). For example, in our example the semimajor axis would have length \( 2R \).