Killing Vector Fields

In order to describe motion in the Schwarzschild spacetime it will be useful to identify certain constants of the motion. We first digress with a more general discussion.

Consider a space(-time) with metric $g_{\mu\nu}$. Under an infinitesimal coordinate transformation $x^\mu \to x'^\mu = x^\mu - \xi^\mu$,

$$g'_{\mu\nu}(x') = \frac{\partial x^\alpha}{\partial x'^\mu}\frac{\partial x^\beta}{\partial x'^\nu} g_{\alpha\beta}(x(x'))$$

$$= \left(\delta^\alpha_\mu + \frac{\partial \xi^\alpha}{\partial x'^\mu}\right)\left(\delta^\beta_\nu + \frac{\partial \xi^\beta}{\partial x'^\nu}\right)\left(g_{\alpha\beta}(x') + \left(\frac{\partial}{\partial x'^\lambda} g_{\alpha\beta}(x')\right)\xi^\lambda + \mathcal{O}(\xi^2)\right)$$

$$= g_{\mu\nu}(x') + g_{\alpha\nu}(x')\frac{\partial \xi^\alpha}{\partial x'^\mu} + g_{\mu\beta}(x')\frac{\partial \xi^\beta}{\partial x'^\nu} + \xi^\lambda \frac{\partial}{\partial x'^\lambda} g_{\mu\nu}(x') + \mathcal{O}(\xi^2)$$

$$= g_{\mu\nu}(x') + \left[\frac{\partial}{\partial x'^\mu}\left(g_{\alpha\nu}\xi^\alpha\right) - \xi^\alpha \frac{\partial}{\partial x'^\mu} g_{\alpha\nu}\right] + \left[\frac{\partial}{\partial x'^\nu}\left(g_{\mu\beta}\xi^\beta\right) - \xi^\beta\frac{\partial}{\partial x'^\nu} g_{\mu\beta}\right]$$

$$+ \xi^\lambda \frac{\partial}{\partial x'^\lambda} g_{\mu\nu} \qquad + \mathcal{O}(\xi^2)$$

Dropping terms of $\mathcal{O}(\xi^2)$

$$= g_{\mu\nu} + \frac{\partial \xi_\nu}{\partial x'^\mu} + \frac{\partial \xi_\mu}{\partial x'^\nu} - \xi_\lambda g^{\alpha\lambda}\left(\frac{\partial g_{\alpha\nu}}{\partial x'^\mu} + \frac{\partial g_{\mu\alpha}}{\partial x'^\nu} - \frac{\partial g_{\mu\nu}}{\partial x'^\alpha}\right)$$

$$= g_{\mu\nu} + \frac{\partial \xi_\nu}{\partial x'^\mu} + \frac{\partial \xi_\mu}{\partial x'^\nu} - 2\,\xi_\lambda \Gamma^\lambda_{\mu\nu}$$

$$\boxed{g'_{\mu\nu} = g_{\mu\nu} + D_\mu \xi_\nu + D_\nu \xi_\mu}$$

A transformation of the coordinates which leaves the form of the metric invariant is called an <u>isometry</u>.
For example, in 3D Euclidean space in Cartesian coordinates x, y, z:

$$ds^2 = dx^2 + dy^2 + dz^2 = g_{ij}\, dx^i dx^j,$$

where $g_{ij} = \delta_{ij}$

Translations of the coordinates leave lengths invariant:

$$x^i \to x'^i = x^i + c^i \quad \text{for constant } c^i$$

– leaves $g_{ij} = \delta_{ij}$.

This corresponds to the translational invariance of the Euclidean geometry, which "looks the same" when translated. Similarly, rotations of the coordinates leave $g_{ij} = \delta_{ij}$, corresponding to the rotational invariance of the Euclidean geometry.

In classical mechanics we are familiar with the consequences of such symmetries: translational invariance → constant momentum, rotational invariance → constant ∠ momentum.

The existence of isometries in a spacetime has similar consequences for freely falling trajectories.

An infinitesimal coordinate transformation specified by $\xi^\mu(x)$ is an isometry if

$$\boxed{D_\mu \xi_\nu + D_\nu \xi_\mu = 0}$$

Killing Equation.

A solution to this equation is called a <u>Killing vector field</u>.

Consider a geodesic with tangent vector $U^\mu = \frac{dx^\mu}{d\tau}$, where τ is the proper time (for a massive object).

Along a geodesic $\frac{DU^\mu}{D\tau} = \frac{dx^\nu}{d\tau} D_\nu U^\mu = \boxed{U^\nu D_\nu U^\mu = 0}$.

Consider $C \equiv \xi_\mu U^\mu$, where ξ^μ is a Killing vector field.

$$\frac{dC}{d\tau} = \frac{DC}{D\tau} = U^\mu \frac{D\xi_\mu}{D\tau} + \xi_\mu \frac{DU^\mu}{D\tau}$$

$$= U^\mu U^\beta D_\beta \xi_\mu + \xi_\mu U^\nu D_\nu U^\mu$$

$$= (0 \text{ by Killing eqn.}) + (0 \text{ by geodesic eqn.})$$

$$\Rightarrow \boxed{\frac{d}{d\tau}\left(\xi_\mu \frac{dx^\mu}{d\tau}\right) = 0}$$

☆ Hence, $\xi_\mu \frac{dx^\mu}{d\tau} = C$ is a constant of the motion

Suppose a metric is independent of one of the coordinates, say x^0, so that $\frac{\partial g_{\mu\nu}}{\partial x^0} = 0$.

Consider the vector field $\xi^\mu = \delta^\mu_0$.

$$\xi_\alpha = g_{\alpha\mu}\xi^\mu = g_{\alpha 0}.$$

$$D_\mu \xi_\nu + D_\nu \xi_\mu = \partial_\mu g_{\nu 0} + \partial_\nu g_{\mu 0} - \underbrace{g^{\alpha\lambda}(\partial_\mu g_{\lambda\nu} + \partial_\nu g_{\mu\lambda} - \partial_\lambda g_{\mu\nu})}_{2\Gamma^\alpha_{\mu\nu}} \underbrace{g_{\alpha 0}}_{\xi_\alpha}$$

Using $g^{\alpha\lambda} g_{\alpha 0} = \delta^\lambda_0$

$$D_\mu \xi_\nu + D_\nu \xi_\mu = \partial_\mu g_{\nu 0} + \partial_\nu g_{\mu 0} - (\partial_\mu g_{0\nu} + \partial_\nu g_{\mu 0} - \cancelto{0}{\partial_0 g_{\mu\nu}})$$
$$= 0.$$

Hence ξ^μ is a Killing vector field.

Motion in the Schwarzschild Spacetime

$$ds^2 = -\left(1-\frac{2GM}{r}\right)dt^2 + \left(1-\frac{2GM}{r}\right)^{-1}dr^2 + r^2\left(d\theta^2 + \sin^2\theta\, d\varphi^2\right)$$
$$= -d\tau^2 \quad \text{(for massive object)}$$

The metric is independent of t and φ, so we can identify two constants of the motion.

t: $$\xi^\mu = \delta^\mu{}_0 \rightarrow \xi_\mu \frac{dx^\mu}{d\tau} = g_{\mu t}\frac{dx^\mu}{d\tau} = -\left(1-\frac{2GM}{r}\right)\frac{dt}{d\tau} \equiv -\tilde{E}$$

$$\boxed{\tilde{E} = \left(1-\frac{2GM}{r}\right)\frac{dt}{d\tau}}$$ is a constant of the motion = "Energy/unit mass."

φ: $$\xi^\mu = \delta^\mu{}_\varphi \rightarrow \xi_\mu \frac{dx^\mu}{d\tau} = g_{\mu\varphi}\frac{dx^\mu}{d\tau} = r^2\sin^2\theta\frac{d\varphi}{d\tau} \equiv \tilde{L}$$

$$\boxed{\tilde{L} = r^2\sin^2\theta\frac{d\varphi}{d\tau}}$$ is a constant of the motion = "Angular momentum/unit mass"

Along the trajectory of a test particle,

$$\left(1-\frac{2GM}{r}\right)\left(\frac{dt}{d\tau}\right)^2 - \left(1-\frac{2GM}{r}\right)^{-1}\left(\frac{dr}{d\tau}\right)^2 - r^2\left(\frac{d\theta}{d\tau}\right)^2 - r^2\sin^2\theta\left(\frac{d\varphi}{d\tau}\right)^2 = K$$

where $K=1$ for massive object (particle)
$K=0$ for massless particle

By the symmetry $\theta \rightarrow \pi-\theta$, if motion begins in the $\theta=\pi/2$ plane, it will remain in that plane. Choose coordinates so that $\theta=\pi/2$.

$$\Rightarrow \left(1-\frac{2GM}{r}\right)\left(\frac{dt}{d\tau}\right)^2 - \left(1-\frac{2GM}{r}\right)^{-1}\left(\frac{dr}{d\tau}\right)^2 - r^2\cancel{\sin^2\theta}^{\,1}\left(\frac{d\varphi}{d\tau}\right)^2 = K$$

$$\frac{\tilde{E}^2}{(1-\frac{2GM}{r})} - \frac{1}{(1-\frac{2GM}{r})}\left(\frac{dr}{d\tau}\right)^2 - \frac{\tilde{L}^2}{r^2} = K$$

$$\boxed{\frac{1}{2}\left(\frac{dr}{d\tau}\right)^2 + \frac{1}{2}\left(1-\frac{2GM}{r}\right)\left(K+\frac{\tilde{L}^2}{r^2}\right) = \frac{1}{2}\tilde{E}^2}$$

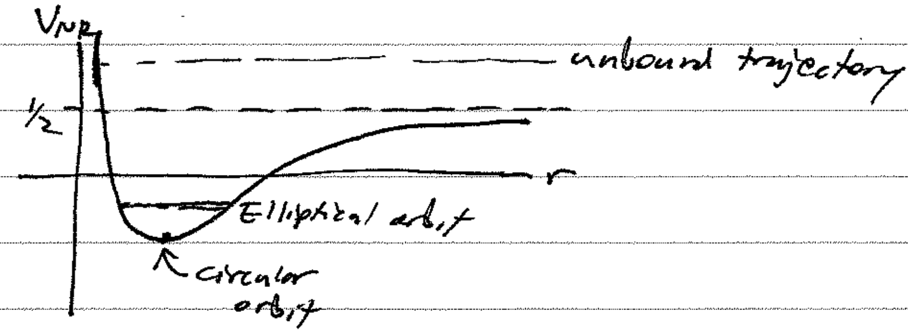
Compare with the Newtonian Limit:

$K=1$, ignore $\frac{1}{r^3}$ term, $\tau \simeq t$

$$\frac{1}{2}\left(\frac{dr}{dt}\right)^2 + \underbrace{\left[\frac{1}{2} - \frac{GM}{r} + \frac{1}{2}\frac{\tilde{L}^2}{r^2}\right]}_{V_{NR}(r)} = \frac{1}{2}\tilde{E}^2$$

$V_{NR}(r)$ nonrelativistic effective potential

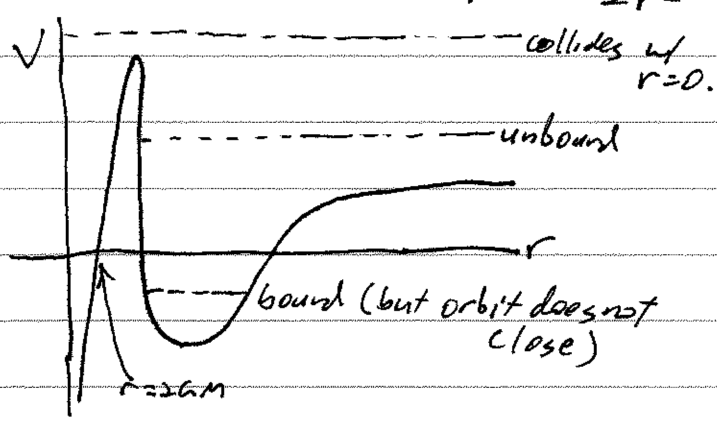
The values of $\tilde{E}^2/2, \tilde{L}^2$ determines the type of trajectory



The $1/r^3$ which we dropped is a correction from GR.

$$V(r) = \frac{1}{2}K - K\frac{GM}{r} + \frac{\tilde{L}^2}{2r^2} - \frac{GM}{r^3}\tilde{L}^2$$

$K=1$:



K=0: