

How to Make a Black Hole

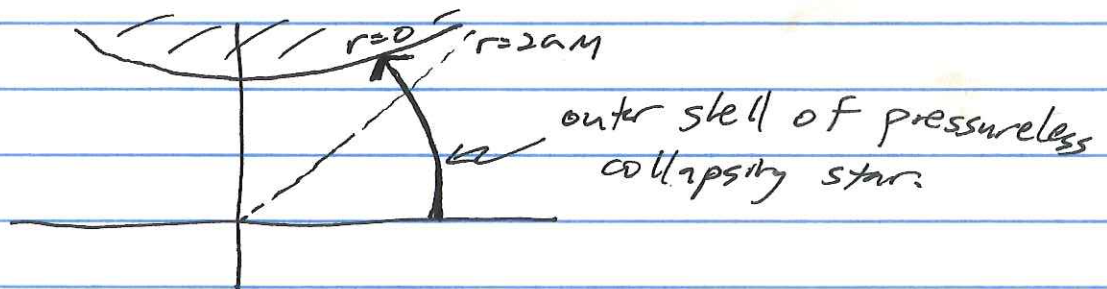
Collapse of Pressureless Dust

\textcircled{R} $r > R: ds^2 = \frac{32 (GM)^3 e^{-r/2GM}}{r} (-dT^2 + dX^2) + r^2 (d\theta^2 + \sin^2\theta d\phi^2)$

Each dust particle on the surface (shell) follows a geodesic toward the center:

$$\begin{cases} r = \frac{R}{2} (1 + \cos \gamma) \\ T = \frac{R}{2} \left(\frac{R}{2GM} \right)^{1/2} (\gamma + \sin \gamma) \end{cases}$$

Kruskal coordinates: $X^2 - T^2 = \left(\frac{r}{2GM} - 1 \right) e^{r/2GM}$



Collapse happens in finite proper time τ , but light signals are delayed so outside observers do not see the shell cross the Schwarzschild radius.

see VII.4

Interior Solutions for Stars

Special Relativity: $T^{\alpha\beta} = \sum_n P_n^\alpha \frac{dx_n^\beta}{dt} \delta^3(\vec{x} - \vec{x}_n(t))$

$T^{00} = \sum_n E_n(t) \delta^3(\vec{x} - \vec{x}_n(t))$ Energy density

$T^{0i} = \sum_n E_n(t) \frac{dx_n^i}{dt} \delta^3(\vec{x} - \vec{x}_n(t))$ Energy Flux in direction i

$T^{ji} = \sum_n P_n^j(t) \frac{dx_n^i}{dt} \delta^3(\vec{x} - \vec{x}_n(t))$ Flux of j -component of momentum in direction i (stress)

Fluids: $N \rightarrow \infty$ limit

Perfect fluid: At each point there is a velocity \vec{v} such that an observer moving w/ velocity \vec{v} observes the fluid as isotropic (the same in every direction).

Local rest frame: $T^{00} = \rho$ Energy density

$T^{0i} = 0$

$T^{ji} = p \delta^{ji}$ ← rotation-invariant
 ↳ pressure

In local rest frame, $U^\alpha = \frac{dx^\alpha}{dt} = (1, 0, 0, 0)$

↳ velocity of fluid element

$$T^{\alpha\beta} = p \gamma^{\alpha\beta} + (p + \rho) U^\alpha U^\beta$$

↳ tensor eqn. → valid in all frames

check: $T^{00} = -p + (p + \rho) = \rho$

$T^{0i} = 0$

$T^{ji} = p \delta^{ji}$ ✓

$$p_n^M = E_n \frac{dx_n^M}{dt} \rightarrow T^{\alpha\beta} = \sum_n \frac{p_n^\alpha p_n^\beta}{E_n} \delta^3(\vec{x} - \vec{x}_n(t))$$

Local rest frame: $T^{00} = \sum_n E_n \delta^3(\vec{x} - \vec{x}_n(t))$

$$\rho = \frac{1}{3}(T^{11} + T^{22} + T^{33}) = \frac{1}{3} \sum_n \frac{\vec{p}_n \cdot \vec{p}_n}{E_n} \delta^3(\vec{x} - \vec{x}_n(t))$$

Massive particles: $E > \frac{p^2}{E} \Rightarrow \rho \geq 3p \geq 0$

Massless particles (photons): $E = \frac{p^2}{E} \Rightarrow \rho = 3p$

General Relativity: $T^{\alpha\beta} = p g^{\alpha\beta} + (p + \rho) U^\alpha U^\beta$

$$U^\alpha = \frac{dx^\alpha}{d\tau}, \quad g_{\alpha\beta} U^\alpha U^\beta = -1$$

Stars: Static, spherically symmetric solution to Einstein's equations

Look for solution of the form

$$ds^2 = -e^{2\phi(r)} dt^2 + e^{2\lambda(r)} dr^2 + r^2 (d\theta^2 + \sin^2\theta d\varphi^2)$$

Fluid elements are at rest $\Rightarrow U^\alpha = (e^{-\phi(r)}, 0, 0, 0)$

$$g_{\alpha\beta} U^\alpha U^\beta = -1 \quad \checkmark$$

$$\begin{cases} T^{\mu\nu} = g^{\mu\nu} p(r) + (p(r) + \rho(r)) U^\mu U^\nu \\ R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R = -8\pi G T^{\mu\nu} \end{cases}$$

Define $m(r)$ by $e^{2\lambda(r)} \equiv \left(1 - \frac{2Gm(r)}{r}\right)^{-1}$

- like Schwarzschild

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Einstein equations:

$$(1) \quad \frac{dm}{dr} = 4\pi r^2 \rho$$

$$(2) \quad \frac{dp}{dr} = -\frac{G\rho m}{r^2} \left(1 + \frac{p}{\rho}\right) \left(1 + \frac{4\pi p r^3}{m}\right) \left(1 - \frac{2Gm}{r}\right)^{-1}$$

- Tolman-Oppenheimer-Volkov eqn.
of hydrostatic equilibrium

$$(3) \quad \frac{d\phi}{dr} = \frac{Gm}{r^2} \left(1 + \frac{4\pi p r^3}{m}\right) \left(1 - \frac{2Gm}{r}\right)^{-1}$$

Solutions:

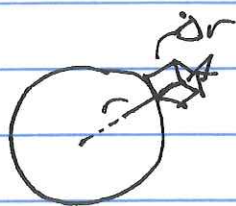
$$(1) \Rightarrow m(r) = 4\pi \int_0^r \rho(r') r'^2 dr' + \text{const.}$$

$$\text{Good behavior at } r=0: e^{2\lambda(r)} = \left(1 - \frac{2Gm(r)}{r}\right)^{-1} \Rightarrow m(0) = 0$$

Assume $2Gm(r) < r$ for all r .

(2): To understand the meaning of the Tolman-Oppenheimer-Volkov equation, consider the Newtonian limit, $\begin{cases} p \ll \rho \\ Gm(r) \ll r \end{cases}$

$$\Rightarrow \frac{dp}{dr} \approx -\frac{G\rho m(r)}{r^2} \quad \text{--- Equation of hydrostatic equilibrium}$$



Gravitational force on infinitesimal box:

$$F_g = \frac{G(A \Delta r \rho(r)) m(r)}{r^2}$$

$$= -(\rho(r + \Delta r)A - \rho(r)A)$$

Divide by Δr : $-\frac{d\rho}{dr} = \frac{G\rho m(r)}{r^2}$ hydrostatic equilibrium

(3) To understand the meaning of eq. (3), consider the Newtonian limit again.

$$\frac{d\phi}{dr} \approx \frac{Gm(r)}{r^2} \rightarrow \phi(r) = G \int_0^r \frac{m(r)}{r^2} dr$$

= Newtonian potential

$$e^{2\phi(r)} \approx 1 + 2\phi(r)$$

Not necessarily in the Newtonian limit!

At the edge of the star $p = 0$

$$r > R : p = \rho = 0 \rightarrow \frac{d\phi}{dr} = \frac{GM}{r^2} \left(1 - \frac{2GM}{r}\right)^{-1}$$

$$\rightarrow e^{2\phi} = 1 - \frac{2GM}{r} \quad \text{Schwarzschild solution}$$

To model a star: (1) Assume an equation of state $p(\rho)$.

(2) Assume a central value for ρ , i.e. $\rho(0) = \rho_c \rightarrow p(0) = p(\rho_c)$

(3) Integrate Einstein eqs¹⁺² to larger r , s.t. $m(0) = 0$.

(4) End integration at r s.t. $p(r) = 0 \rightarrow r = R$.

(5) Integrate Einstein eq. 3 to find $\phi(r)$.

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Example: Incompressible Fluid

$\rho = \text{constant}$, indep. of P

$$\rho(r) = \begin{cases} \rho_0 & r \leq R \\ 0 & r > R \end{cases}$$

$$m(r) = \frac{4}{3} \pi \rho_0 r^3$$

$$\frac{dp}{dr} = -G \frac{4}{3} \pi \rho_0^2 r \left(1 + \frac{P}{\rho_0}\right) \left(1 + \frac{3P}{\rho_0}\right) \left(1 - \frac{8}{3} \pi \rho_0 G r^2\right)^{1/2}$$

Integrate:

$$p(r) = \rho_0 \left[\frac{\left(1 - \frac{2GM}{R}\right)^{1/2} - \left(1 - \frac{2GM r^2}{R^3}\right)^{1/2}}{\left(1 - \frac{2GM r^2}{R^3}\right)^{1/2} - 3 \left(1 - \frac{2GM}{R}\right)^{1/2}} \right]$$

$p(R) = 0 \rightarrow$ defines R (edge of star)

$r=0$: Denominator of $p(r)$ would vanish if

$$1 - 3 \left(1 - \frac{2GM}{R}\right)^{1/2} = 0$$

$$\rightarrow \frac{GM}{R} = \frac{4}{9}$$

If $\frac{GM}{R} > \frac{4}{9}$ then $p(r) \rightarrow \infty$ for some $r > 0$

-unphysical

$$\rightarrow \boxed{GM < \frac{4R}{9}}$$

Bound on mass of star

In fact, the bound $GM < \frac{4R}{a}$ follows more generally as long as $\rho(r) > 0$ and $\frac{d\rho}{dr} < 0$.

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White Dwarfs - Old star burns its fuel, cools and contracts.

Low temperature $T \rightarrow$ electrons freeze into lowest available energy levels.

Number of electrons/unit volume:

$$n = 2 \cdot \frac{4\pi}{(2\pi\hbar)^3} \int_0^{k_F} k^2 dk$$

\leftarrow Fermi momentum
 \uparrow spins \uparrow from density of states

$$\Rightarrow \boxed{n = \frac{k_F^3}{3\pi^2\hbar^3}}$$

Mass density mainly from nucleons:

$$\rho = n m_N \mu$$

\uparrow avg. nucleon mass \leftarrow # nucleons/electron

Iron: $\mu = \frac{56}{26}$

$$\Rightarrow \boxed{k_F = \hbar \left(\frac{3\pi^2 \rho}{m_N \mu} \right)^{1/3}}$$

Electron pressure:

$$p = \frac{8\pi}{3(2\pi\hbar)^3} \int_0^{k_F} \frac{k^2}{\sqrt{k^2 + m_e^2 c^2}} k^2 dk$$

$\uparrow \frac{p^2}{E}$ (from energy-momentum tensor T_{ij})

If $k_F \gg m_e$:

$$p \approx \frac{8\pi k_F^4}{12(2\pi\hbar)^3} = \frac{\hbar}{12\pi^2} \left(\frac{3\pi^2}{m_N \mu} \right)^{4/3} \rho^{4/3}$$

$$\equiv K \rho^{4/3} \quad \text{Equation of state}$$

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- 1) Integrate the equation for hydrostatic equilibrium with this equation of state gives $p(r)$.
- 2) pressure vanishes \rightarrow radius of star R .
- 3) $M = \int_0^R 4\pi r'^2 \rho(r') dr'$

$$\Rightarrow M = \frac{1}{2} (3\pi)^{4/3} (2.01824) \left(\frac{\hbar^{3/2} c^{3/2}}{G^{3/2} m_N^2 \mu^2} \right)$$

$$= \boxed{5.87 \mu^{-2} M_\odot \equiv M_C}$$

For smaller k_F , $M < M_C$. Hence, white dwarf stars have mass $< M_C \equiv$ Chandrasekhar limit.

In fact, for $k_F \approx 5 m_e$, it is energetically favorable for electrons to be captured by nuclei, converting protons to neutrons.
 $\Rightarrow M < 1.2 M_\odot$, close to Chandrasekhar limit of $\mu = \frac{56}{26}$