Weinberg 12.4

An Action Principle for Gravity

The gravitational action should be invariant under coordinate transformations and should be composed of terms with two derivatives of the metric and its inverse.

The invariant volume element is $\sqrt{|g|}\, d^4x$, and the only scalar that suits the bill is R. Hence, we will study the equations that follow by varying the action $S = S_M + S_G$, with S_M the matter action, and

$$\boxed{S_G = -\frac{1}{16\pi G}\int d^4x \sqrt{|g|}\, R}$$ Einstein-Hilbert action

Taking $g_{\mu\nu} \to g_{\mu\nu} + \delta g_{\mu\nu}$, using $R = g^{\mu\nu} R_{\mu\nu}$,

$$\delta\left(\sqrt{|g|}\, R\right) = \sqrt{|g|}\, R_{\mu\nu}\, \delta g^{\mu\nu} + R\, \delta\sqrt{|g|} + \sqrt{|g|}\, g^{\mu\nu}\, \delta R_{\mu\nu}.$$

From the definition of $R_{\mu\nu}$ in terms of $\Gamma^\lambda_{\mu\nu}$, and $\Gamma^\lambda_{\mu\nu}$ in terms of $g_{\mu\nu}$, it is straightforward to show the Palatini identity:

$$\boxed{\delta R_{\mu\nu} = \left(\delta\Gamma^\lambda_{\mu\lambda}\right)_{;\nu} - \left(\delta\Gamma^\lambda_{\mu\nu}\right)_{;\lambda}}$$

Hence $\sqrt{|g|}\, g^{\mu\nu}\, \delta R_{\mu\nu} = \sqrt{|g|}\left[\left(g^{\mu\nu}\delta\Gamma^\lambda_{\mu\lambda}\right)_{;\nu} - \left(g^{\mu\nu}\delta\Gamma^\lambda_{\mu\nu}\right)_{;\lambda}\right]$

Using $V^\mu{}_{;\mu} = \frac{1}{\sqrt{|g|}}\, \partial_\mu\left(\sqrt{|g|}\, V^\mu\right)$,

$$\sqrt{|g|}\, g^{\mu\nu} \delta R_{\mu\nu} = \partial_\nu\left(\sqrt{|g|}\, g^{\mu\nu} \delta \Gamma^\lambda_{\mu\lambda}\right) - \partial_\lambda\left(\sqrt{|g|}\, g^{\mu\nu} \delta \Gamma^\lambda_{\mu\nu}\right)$$

Hence, $\int d^4x \sqrt{|g|}\, g^{\mu\nu} \delta R_{\mu\nu} = 0.$

Using $\delta \ln \det M = \operatorname{Tr} M^{-1} \delta M$ with $M = g_{\mu\nu}$,

$$\boxed{\delta \sqrt{|g|} = \frac{1}{2} \sqrt{|g|}\, g^{\mu\nu} \delta g_{\mu\nu}}$$

Using $\delta(g^{\mu\nu} g_{\nu\lambda}) = 0$, $\boxed{\delta g^{\mu\nu} = -g^{\mu\rho} g^{\nu\sigma} \delta g_{\rho\sigma}}$

Hence, $\boxed{\delta S_G = \frac{1}{16\pi G} \int d^4x \sqrt{|g|} \left[R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R\right] \delta g_{\mu\nu}}$

The energy-momentum tensor for matter may be defined in terms of the variation of the matter action with respect to the metric:

$$\boxed{\delta S_M = \frac{1}{2} \int d^4x \sqrt{|g|}\, T^{\mu\nu} \delta g_{\mu\nu}}$$

Hence,
$$\begin{aligned}\delta S &= \delta S_G + \delta S_M \\ &= \frac{1}{16\pi G} \int d^4x \sqrt{|g|}\, \delta g_{\mu\nu} \left[R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R + 8\pi G T^{\mu\nu}\right]\end{aligned}$$

$\delta S = 0$ gives the equation of motion, which we recognize as the Einstein equation:

$$\boxed{R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R = -8\pi G T^{\mu\nu}}$$

The Matter Action

Example: Scalar field

Suppose a scalar field $\phi(x)$ is described by the action

$$S_m = \int d^4x \sqrt{|g|} \underbrace{\left\{ -\frac{1}{2} \partial_\mu \phi \, \partial_\nu \phi \, g^{\mu\nu} - V(\phi) \right\}}_{\mathcal{L}}$$

where the potential $V(\phi)$ depends on ϕ but not its derivatives.

In classical mechanics, stationarizing the action under variation of trajectories $q(t)$ leads to the Euler-Lagrange equation,

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{q}} = \frac{\partial L}{\partial q}.$$

Similarly, stationarizing S_m with respect to variations of $\phi(x)$ leads to the equation of motion for $\phi(x)$,

$$\frac{\partial}{\partial x^\mu} \left(\frac{\partial}{\partial(\partial_\mu \phi)} \left(\sqrt{|g|} \mathcal{L} \right) \right) = \frac{\partial}{\partial \phi} \left(\sqrt{|g|} \mathcal{L} \right), \text{ or}$$

$$-\partial_\mu \left(\sqrt{|g|} (\partial_\nu \phi) g^{\mu\nu} \right) = -\sqrt{|g|} \frac{\partial V}{\partial \phi}, \text{ or}$$

$$\frac{1}{\sqrt{|g|}} \partial_\mu \left(\sqrt{|g|} (\partial_\nu \phi) g^{\mu\nu} \right) = \frac{\partial V}{\partial \phi}, \text{ or}$$

$$\boxed{D_\mu D^\mu \phi = \frac{\partial V}{\partial \phi}}.$$

If $V = 0$ this is the wave equation for ϕ in curved spacetime.

The energy-momentum tensor for the scalar field is

$$T^{\mu\nu} = \frac{2}{\sqrt{|g|}} \frac{\delta S_m}{\delta g_{\mu\nu}} \quad , \text{ or}$$

$$T_{\mu\nu} = -\frac{2}{\sqrt{|g|}} \frac{\delta S_m}{\delta g^{\mu\nu}} .$$

Using $\delta \sqrt{|g|} = \frac{1}{2} \sqrt{|g|}\, g^{\mu\nu} \delta g_{\mu\nu} = -\frac{1}{2} \sqrt{|g|}\, g_{\mu\nu} \delta g^{\mu\nu}$, we have

$$\boxed{T_{\mu\nu} = \partial_\mu \phi\, \partial_\nu \phi - g_{\mu\nu} \mathcal{L}}$$

The Vielbein Formalism

In order to turn a special-relativistic equation into a general-relativistic equation including gravity, a minimal prescription is to replace $\eta_{\mu\nu}$ with $g_{\mu\nu}$, and derivatives ∂_μ with covariant derivatives D_μ. This works for equations involving tensors, but is not as straightforward for spinors.

Under a rotation by angle θ about axis $\hat{n}$, a 2-component spinor transforms as

$$\chi \xrightarrow{R(\theta,\hat{n})} e^{\frac{i\vec{\sigma}\cdot\hat{n}\theta}{2}}\chi,$$

where $\vec{\sigma} = (\sigma_1, \sigma_2, \sigma_3)$ are the Pauli sigma matrices.

There are related spinor representations of the group of Lorentz transformations, $SO(3,1)$.

However, there are no spinor representations of the group of general linear transformations appropriate for general relativity. So how might we include gravitation in the Dirac equation for the relativistic electron, for example?

This puzzle motivates a different approach to determine the effects of gravitation, an approach still grounded in the Principle of Equivalence.

At each point in spacetime there are locally inertial coordinates ξ^a. In the neighborhood of that point the spacetime metric takes the special relativistic form,

$$ds^2 = \eta_{ab}\, d\xi^a\, d\xi^b.$$

In a general coordinate system, the metric becomes

$$ds^2 = \eta_{ab} \frac{\partial \xi^a}{\partial x^\mu} \frac{\partial \xi^b}{\partial x^\nu} dx^\mu dx^\nu$$

$$\equiv g_{\mu\nu}\, dx^\mu dx^\nu.$$

Hence, $g_{\mu\nu} = \eta_{ab} \dfrac{\partial \xi^a}{\partial x^\mu} \dfrac{\partial \xi^b}{\partial x^\nu}$

$$\boxed{g_{\mu\nu} \equiv \eta_{ab}\, e_\mu{}^a\, e_\nu{}^b}$$

where $e_\mu{}^a \equiv \dfrac{\partial \xi^a}{\partial x^\mu}$ is called a tetrad or

vierbein in four spacetime dimensions, or a vielbein more generally.

We think of $e_\mu{}^a$ as a set of 4 covariant vector fields $e_\mu{}^0$, $e_\mu{}^1$, $e_\mu{}^2$, and $e_\mu{}^3$.

Given a contravariant 4-vector V^μ, the vierbein transforms the vector to locally inertial coordinates:

$$V^a \equiv e_\mu{}^a V^\mu,$$

which we think of as a set of four scalars V^0, V^1, V^2, V^3.

We can construct an inverse vielbein which transforms covariant vectors to locally inertial coordinates at a point:

$$g_{\mu\sigma} = \eta_{ab}\, e_\mu{}^a e_\sigma{}^b$$

$$\delta_\mu^\nu = g^{\nu\sigma} g_{\mu\sigma} = e_\mu{}^a \underbrace{e_\sigma{}^b \eta_{ab} g^{\nu\sigma}}_{(e^{-1})_a{}^\nu \equiv e_a{}^\nu}$$

(note the placement of Lorentz index a and general coordinate index ν to distinguish $e_\nu{}^a$ and $e_a{}^\nu$.)

Because $e_a{}^\nu$ is the inverse of $e_\mu{}^a$, we also have

$$e_a{}^\mu e_\mu{}^b = \delta_a^b$$

$$\left(e_\sigma{}^c \eta_{ac} g^{\mu\sigma}\right) e_\mu{}^b = \delta_a^b$$

$$\times \eta^{ad}: \quad \boxed{e_\sigma{}^d e_\mu{}^b g^{\mu\sigma} = \eta^{db}}$$

Similarly, $\boxed{e_a{}^\mu e_b{}^\nu g_{\mu\nu} = \eta_{ab}}$ (Exercise)

By always referring to locally inertial coordinates, tensors are reduced to sets of scalars under coordinate transformations. We can also refer spinors to locally inertial coordinates, and the special-relativistic description of spinors is again relevant.

In this formalism, equations describing interactions with gravitation need to satisfy two conditions:

1) The equation is generally covariant, with fields treated as scalars <u>except</u> the vielbein $e_\mu{}^a$.

2) The equation is covariant under independent Lorentz transformations at each point.
For example $V^a(x) \to \Lambda^a{}_b(x)\, V^b(x)$, etc.

The second requirement demands a new type of covariant derivative $D_a = e_a{}^\mu \Big(\underbrace{\frac{\partial}{\partial x^\mu} + \Gamma_\mu(x)}_{\equiv D_\mu} \Big)$

where $\Gamma_\mu(x)$ is a matrix depending on what type of tensor or spinor field it acts on, defined so that D_μ acting on an object transforms under local Lorentz transformations the same way as the object, canceling nonhomogeneous terms like $\frac{\partial}{\partial x^\mu}\Lambda^a{}_b(x)$.

$\Gamma_\mu(x)$ is called the <u>spin connection</u>.

Example: Covariant derivative of covariant vector:

$$(D_\mu V)_c = \partial_\mu V_c + (\Gamma_\mu)^d{}_c V_d \text{ , where}$$

Γ_μ acting on a covariant vector has the form

$$(\Gamma_\mu)^d{}_c = e_a^\lambda \left(\partial_\mu e_{\lambda b} - \Gamma^\sigma_{\mu\lambda} e_{\sigma b}\right)(\Sigma^{ab})_c{}^d$$

Christoffel symbols

where $e_{\lambda b} \equiv \eta_{bc}\, e_\lambda{}^c$ and

$$(\Sigma^{ab})_c{}^d = \frac{1}{2}\left(\delta^a_c\, \eta^{bd} - \delta_c^b\, \eta^{ad}\right)$$

(= Lorentz generator in vector representation)

After some tedious algebra, one can check that
1) $(D_\mu V)_c$ transforms as a vector under local Lorentz transformations, and
2) $(D_\mu V)_c$ agrees with the usual covariant derivative:

$$(D_\mu V)_\nu \equiv e_\nu{}^c (D_\mu V)_c = \partial_\mu V_\nu - \Gamma^\sigma_{\mu\nu} V_\sigma$$

In d spacetime dimensions, it would seem that $e_\mu{}^a$ has $d \times d = d^2$ independent components. However, $\frac{d(d-1)}{2}$ local Lorentz transformations can restrict the same # of components.
Hence, there are $d^2 - \frac{d(d-1)}{2} = \frac{d(d+1)}{2}$ independent components of the vielbein up to local Lorentz transfs, the same # of components as in $g_{\mu\nu}$.

Appendix:

Spin Connection vs Christoffel Symbols

$$(D_\mu V)_C = \partial_\mu V_C + (\Gamma_\mu)^D_C V_D \quad , \quad V = \text{Lorentz vector (covariant)}$$

Γ_μ = Spin Connection

$$(D_\mu V)_C = \partial_\mu V_C + \underbrace{e^\lambda{}_A(\partial_\mu e_{\lambda B} - \Gamma^\sigma_{\mu\lambda} e_{\sigma B})}_{\omega_{\mu AB}} (\Sigma^{AB})_C{}^D V_D$$

Christoffel symbols

$$(\Sigma^{AB})_C{}^D = \frac{1}{2}\left(\delta^A_C \eta^{BD} - \delta^B_C \eta^{AD}\right)$$

= Lorentz generator in vector representation

$$(D_\mu V)_C = \partial_\mu V_C + \frac{1}{2}\left(e^\lambda{}_C \partial_\mu e_\lambda{}^D - e^{\lambda D}\partial_\mu e_{\lambda C} - \Gamma^\sigma_{\mu\lambda} e^\lambda{}_C e_\sigma{}^D + \Gamma^\sigma_{\mu\lambda} e^{\lambda D} e_{\sigma C}\right) V_D$$

$$= \partial_\mu V_C + \frac{1}{2}\left(e^\lambda{}_C \partial_\mu e_\lambda{}^D - e^{\lambda D}\partial_\mu e_{\lambda C} - \Gamma^\sigma_{\mu\lambda} e^\lambda{}_C e_\sigma{}^D + \frac{1}{2} e^{\lambda D} e_{\sigma C} g^{\sigma\gamma}\left(\partial_\mu g_{\lambda\gamma} + \partial_\lambda g_{\mu\gamma} - \partial_\gamma g_{\mu\lambda}\right)\right) V_D$$

$$= \partial_\mu V_C + \frac{1}{2}\left(e^\lambda{}_C \partial_\mu e_\lambda{}^D - e^{\lambda D}\partial_\mu e_{\lambda C} - \Gamma^\sigma_{\mu\lambda} e^\lambda{}_C e_\sigma{}^D + \frac{1}{2} e^{\lambda D} e^\gamma{}_C \left(\partial_\mu g_{\lambda\gamma} + \partial_\lambda g_{\mu\gamma} - \partial_\gamma g_{\mu\lambda}\right)\right) V_D$$

$$e^C_\nu (D_\mu V)_C = e_\nu{}^C \partial_\mu V_C + \frac{1}{2}\left(\partial_\mu e_\nu{}^D - e^{\lambda D} e_\nu{}^C \partial_\mu e_{\lambda C} - \Gamma^\sigma_{\mu\nu} e_\sigma{}^D + \frac{1}{2} e^{\lambda D}\left(\partial_\mu g_{\lambda\nu} + \partial_\lambda g_{\mu\nu} - \partial_\nu g_{\mu\lambda}\right)\right) V_D$$

$$e_\nu{}^C (D_\mu V)_C = e_\nu{}^C \partial_\mu V_C + \frac{1}{2}\Big(\partial_\mu e_\nu{}^D - e^{\lambda D}(\partial_\mu g_{\nu\lambda} - e_{\lambda C}\partial_\mu e_\nu{}^C)$$

$$- \Gamma^\sigma_{\mu\nu} e_\sigma{}^D + \frac{1}{2} e^{\lambda D}(\partial_\mu g_{\nu\lambda} + \partial_\lambda g_{\mu\nu} - \partial_\nu g_{\mu\lambda})\Big) V_D$$

$$= e_\nu{}^C \partial_\mu V_C + \frac{1}{2}\Big(\partial_\mu e_\nu{}^D + \partial_\mu e_\nu{}^D - \Gamma^\sigma_{\mu\nu} e_\sigma{}^D$$

$$+ \frac{1}{2} e^{\lambda D}(-\partial_\mu g_{\nu\lambda} + \partial_\lambda g_{\mu\nu} - \partial_\nu g_{\mu\lambda})\Big) V_D$$

$$= e_\nu{}^C \partial_\mu V_C + (\partial_\mu e_\nu{}^D) V_D - \Gamma^\sigma_{\mu\nu} e_\sigma{}^D V_D$$

$$= \partial_\mu V_\nu - \cancel{V_C \partial_\mu e_\nu{}^C} + \cancel{(\partial_\mu e_\nu{}^D) V_D} - \Gamma^\sigma_{\mu\nu} V_\sigma$$

$$\boxed{(D_\mu V)_\nu \equiv e_\nu{}^C (D_\mu V)_C = \partial_\mu V_\nu - \Gamma^\sigma_{\mu\nu} V_\sigma}$$