Physics 772, Spring 2009
Problem Set 2 Due Tuesday, March 3.

1. *Massive electrodynamics*

As we have discussed, the W and Z bosons obtain their mass through the Higgs mechanism. The quantum theory of vector fields requires gauge invariance for renormalizability, but it is valuable to consider briefly the most general free theory of massive vector fields so that we can compare with the gauge-invariant theory.

The most general Lagrangian density for the vector field including terms quadratic in the field with at most two derivatives is (up to the addition of total derivatives):

$$\mathcal{L} = -\frac{1}{2}\,\partial_\mu A_\nu\,\partial^\mu A^\nu + b\,\partial_\mu A_\nu \partial^\nu A^\mu + c\,A_\mu A^\mu.$$

(a) What are the Euler-Lagrange equations for this theory?

(b) Assume a plane wave solution of the form $A^\mu(x) = \varepsilon^\mu(\mathbf{k})e^{-ik\cdot x}$. What are the Euler-Lagrange equations in terms of ε^μ and k^μ?

(A^μ is real, but as usual we describe two solutions at once – the real and imaginary parts of the plane wave. We can do this because the Euler-Lagrange equations are linear in this theory.)

(c) **Longitudinal mode:** Assume $\varepsilon^\mu(\mathbf{k}) \propto k^\mu$. What are the Euler-Lagrange equations for this *ansatz* in terms of ε^μ and k^μ? Define $k_\mu k^\mu \equiv m_L^2$. What is the longitudinal mass m_L in terms of the parameters b and c in the Lagrangian?

(d) **Transverse modes:** Repeat part (c) assuming that $\varepsilon_\mu k^\mu = 0$. This time define $k_\mu k^\mu \equiv m_T^2$. What is m_T?

(e) The longitudinal mode will not propagate if $m_L \to \infty$. What choice of b accomplishes this? Make that choice and rewrite $\mathcal{L}$ in terms of A^μ, m_T, and $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$.

This is the Proca Lagrangian of massive electrodynamics. You should recover Maxwell's theory if you take $m_T \to 0$.

(f) Consider the Proca Lagrangian you have just derived. If $m_T \neq 0$ then the action is not gauge invariant. Show that the Lorenz gauge condition $\partial_\mu A^\mu = 0$ follows from the equations of motion as long as $m_T \neq 0$.

(g) Using the functional integral formalism, show that the propagator for the Proca field takes the form

$$\langle 0| T\left(A_\mu(x) A_\nu(y)\right)|0\rangle = \int \frac{d^4k}{(2\pi)^4} \frac{-i\left(g_{\mu\nu} - \frac{k_\mu k_\nu}{m_T^2}\right)}{k^2 - m_T^2 + i\epsilon} e^{-ik\cdot(x-y)}.$$

This theory is not gauge invariant, so gauge fixing is not required (or appropriate).

(h) **Decoupling of the Helicity-0 Mode:** In momentum space the current conservation law becomes $k_\mu \tilde{J}^\mu(k) = 0$. The expansion of $A_\mu(x)$ in plane waves includes modes with helicity ± 1 and 0. Consider a mode with spatial momentum in the $\mathbf{x}^3$ direction: $k^\mu = (\sqrt{k_3^2 + m_T^2}, 0, 0, k_3)$. The polarization of the normalized helicity-0 mode is then $\varepsilon_\mu^{(0)}(k_3) = \frac{1}{m_T}(k_3, 0, 0, -\sqrt{k_3^2 + m_T^2})$.

By expanding in powers of m_T/k_3 show that in the massless photon limit, $\varepsilon_\mu^{(0)}(k_3)\tilde{J}^\mu(-\omega_{k_3}, -k_3) \to 0$. In other words, the coupling of the helicity-0 mode to the conserved current vanishes as the photon mass goes to zero.

Comments: 1) In the literature what we have called the helicity-0 mode is usually referred to as longitudinal. This is just a warning to help avoid future confusion. 2) As a consequence of the decoupling of the helicity-0 mode, the massless limit of massive electrodynamics is equivalent to massless electrodynamics. One could do the same for the theory of a spin-2 field, *i.e.* a symmetric tensor field $h_{\mu\nu}$ coupled to a conserved stress-energy tensor. This is the theory of linearized massive gravity. In the massless limit of this theory the helicity ± 2 modes remain coupled, and the helicity ± 1 modes decouple. However, the helicity-0 mode remains coupled (to the trace of the stress tensor). Hence, at the linearized level the massless limit of massive gravity is not equivalent to massless gravity. This fact is known as the Van Dam-Veltman-Zakharov discontinuity, and is the source of some confusion and discussion in the gravity literature.