Physics 772, Spring 2009
Problem Set 1
Due Thursday, Feb 19.

1. *Lie Groups*

a) Show that if the generators of a Lie group satisfy the Lie algebra,

$$\left[T^a, T^b\right] = i\, f^{abc}\, T^c,$$

then the generators in the adjoint representation, defined by

$$(T^a)_{bc} = -i\, f^{abc},$$

satisfy the Lie algebra.

b) Show that $T^2 = \sum_a T^a T^a$ commutes with each of the group generators. As a consequence, $(T^2)_{ij} = C_2(r)\, \delta_{ij}$, where the constant $C_2(r)$ is called the **quadratic Casimir** of the representation r.

c) Suppose the generators are normalized so that

$$\mathrm{Tr}\, T^a T^b = \mu_r\, \delta^{ab},$$

where the constant μ_r depends on the representation r. Suppose the generators in the representation r are $n_r \times n_r$ matrices. Show that

$$n_r\, C_2(r) = n_{\text{adjoint}}\, \mu_r.$$

d) Suppose the generators of SU(N) in the fundamental representation are normalized by

$$\mathrm{Tr}\, T^a T^b = \frac{1}{2} \delta^{ab}.$$

Calculate the quadratic Casimir C_2 in this representation.

2. *Gauge theory*

a) Under a gauge transformation, the gauge fields $A_\mu = A^a_\mu T^a$ transform as

$$A_\mu \to U A_\mu U^\dagger - \frac{i}{e} U \partial_\mu U^\dagger.$$

By considering an infinitessimal constant transformation $U = \exp\left[i \sum_a \theta^a T^a\right]$, with $\theta^a \ll 1$, show that the gauge fields A^a_μ transform in the adjoint representation of the gauge group.

b) Assume a set of Dirac spinor fields Ψ_{Ij}, $I = 1, \ldots, n_{r_1}$, $j = 1, \ldots, n_{r_2}$, transforms in an $(n_{r_1} \times n_{r_2})$-dimensional representation of a product gauge group SU$(N_1)\times$ SU(N_2). The generators of the group in this representation take the form $T^A_{IJ} \times \mathbf{1}$ and $\mathbf{1} \times t^a_{ij}$, with $A = 1, \ldots, N_1^2 - 1$ and $a = 1, \ldots, N_2^2 - 1$, and the gauge couplings are e_1 and e_2.

Write the form of the gauge-invariant Lagrangian, including appropriate gauge fields and assuming the Dirac spinor fields are minimally coupled.

c) In part (b), assume the gauge group is SU$(N)\times$ SU(N), and the Dirac spinors transform in a representation r of the first SU(N) and the conjugate representation $\overline{r}$ of the second SU(N).

Show that the following Lagrangian density is gauge invariant:

$$\mathcal{L} = \overline{\Psi}_{Ii}\, \gamma^\mu \left(i\partial_\mu \Psi_{Ii} - e_1 A^A_\mu T^A_{IJ} \Psi_{Ji} + e_2 \Psi_{Ij} B^b_\mu T^b_{ji} \right),$$

where A^A_μ and B^b_μ are the gauge fields associated with the two SU(N) gauge group factors, and e_1, e_2 the respective gauge couplings.

3. *Functional Integral Quantization*

a) Using the functional integral for a free complex scalar field ϕ with mass m, evaluate the following correlation functions from the functional integral:

$$\langle 0|\phi(x)|0\rangle, \ \langle 0|T\left[\phi(x)\phi(y)\right]|0\rangle, \ \langle 0|T\left[\phi(x)\overline{\phi}(y)\right]|0\rangle.$$

b) Consider a general quadratic action for a quantum mechanical system described by a coordinate $q(t)$,

$$S[q] = \int dt \left[a(t)\dot{q}^2 + b(t)\dot{q} + c(t)q\,\dot{q} + d(t)q + e(t)q^2 + f(t) \right].$$

Show that the transition amplitude $\langle q_f, t_f | q_i, t_i\rangle$ takes the form

$$\langle q_f, t_f | q_i, t_i\rangle = F(t_f, t_i)\, \exp\left[iS_c(q_f, t_f; q_i, t_i) \right],$$

where $F(t_f, t_i)$ is independent of q_i and q_f, and S_c is the action for the classical trajectory which solves the classical equation of motion for $q(t)$.

c) For a free quantum mechanical particle with mass m in one spatial dimension, use the functional integral to show that the transition amplitude takes the form

$$\langle q_f, t_f | q_i, t_i\rangle = \left[\frac{m}{2\pi i(t_f - t_i)} \right] \exp\left[\frac{i\,m}{2} \frac{(q_f - q_i)^2}{t_f - t_i} \right].$$