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Self Energies (for scalar fields)

So far we have defined our renormalized perturbation theory such that the physical masses appear in the free part of the Lagrangian, but we should define what we mean by the physical mass to make this meaningful.

In the free theory, mass appears in the propagator as,

$$\langle 0|T\phi(x)\phi(y)|0\rangle = \int \frac{d^4k}{(2\pi)^4} \frac{i\, e^{-ik\cdot(x-y)}}{k^2-m^2+i\epsilon}$$

i.e., $$\boxed{\int d^4x\, e^{ik\cdot x}\langle 0|T\phi(x)\phi(0)|0\rangle = \frac{i}{k^2-m^2+i\epsilon}}$$

(for a real scalar field)

Mass appears as the location of the pole in the propagator.

In fact, this will be our definition of the physical mass in the interacting theory:

$\boxed{\text{The physical mass is the location of the pole in the propagator, } \int d^4x\, e^{ik\cdot x}\langle 0|T\phi(x)\phi(0)|0\rangle.}$

This is in fact a quite sensible definition:

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Recall the completeness relation for 1-particle states:

$$\mathbb{1}_{\text{1-particle}} = \int \frac{d^3k}{(2\pi)^3} \frac{1}{2\omega_{\vec{k}}} |\vec{k}\rangle\langle\vec{k}|$$

$|\vec{k}\rangle$ is an eigenstate of the momentum operator $\vec{P}$ with momentum $\vec{k}$. Since $\vec{P}$ commutes w/ the Hamiltonian H, $|\vec{k}\rangle$ is chosen to also be an eigenstate of H w/ eigenvalue $\boxed{\omega_{\vec{k}} = \sqrt{\vec{k}^2+m^2}}$, where m is the physical mass. Since $(H, \vec{P})$ transform as a 4-vector, so do their eigenvalues. Hence, boosting the state $|\vec{k}\rangle$ yields a state w/ boosted momentum $\vec{k}'$ and energy $\omega_{\vec{k}'} = \sqrt{\vec{k}'^2+m^2}$. Lorentz invariance forces the 1-particle energies and momenta to lie on a hyperboloid, the <u>mass shell</u>: $\omega^2 - \vec{k}^2 = m^2$.

Now consider the propagator. Assume $x^0 > 0$. Then,

$$\begin{aligned}\langle 0|T\,\phi(x)\,\phi(0)|0\rangle &= \langle 0|\phi(x)\,\phi(0)|0\rangle \\ &= \langle 0|\phi(x)|0\rangle\langle 0|\phi(0)|0\rangle + \langle 0|\phi(x)\,\mathbb{1}_{\text{1-particle}}\,\phi(0)|0\rangle \\ &\quad + \cdots\end{aligned}$$

^ from $\mathbb{1}_{\substack{\text{not vac or}\\ \text{1-particle}}}$ inserted between fields.

Consider the contribution from 1-particle states:

$$\langle 0|\phi(x)\,\mathbb{1}_{\text{1-particle}}\,\phi(0)|0\rangle = \int \frac{d^3k}{(2\pi)^3\, 2\omega_k} \langle 0|\phi(x)|k\rangle\langle k|\phi(0)|0\rangle$$

$$= \int \frac{d^3k}{(2\pi)^3\, 2\omega_k} \langle 0|e^{iP\cdot x}\,\phi(0)\,e^{-iP\cdot x}|k\rangle\langle k|\phi(0)|0\rangle$$

$\langle 0 | \phi(x)\, \mathbb{1}_{1\text{-particle}}\, \phi(0) | 0 \rangle$

$$= \int \frac{d^3k}{(2\pi)^3 2\omega_k} |\langle 0 | \phi(0) | \vec{k} \rangle|^2 e^{-ik\cdot x}$$

$$= \int \frac{d^3k}{(2\pi)^3 2\omega_k} e^{-ik\cdot x} |\langle 0 | \phi(0) | \vec{0} \rangle|^2$$

vacuum ↗ ↑ 1-particle state w/ $\vec{k} = \vec{0}$.

$$= \int \frac{d^4k}{(2\pi)^4} \frac{i e^{-ik\cdot x}}{k^2 - m^2 + i\epsilon} |\langle 0 | \phi(0) | \vec{0} \rangle|^2$$

Fourier transforming, $\int d^4x \langle 0 | \phi(x) \phi(0) | 0 \rangle e^{ik\cdot x}$

$$= \frac{i}{k^2 - m^2 + i\epsilon} |\langle 0 | \phi(0) | \vec{0} \rangle|^2 + \text{contribution from non-1 particle states.}$$

The 1-particle states gave rise to a <u>pole</u> in the propagator at the physical mass m. The contributions from the other states will not eliminate this pole. Hence, we can use the location of the pole as the <u>definition</u> of the physical mass of the particle w/ nonvanishing $\langle 0 | \phi(0) | \vec{k} \rangle$.

Note that if we had instead calculated w/ the renormalized field $\tilde{\Phi}(x)$ s.t. $\langle 0 | \tilde{\Phi}(0) | k \rangle = 1$, then the <u>residue</u> of the pole would be its free value, i:

$$\boxed{\int d^4x \langle 0 | \tilde{\Phi}(x) \tilde{\Phi}(0) | 0 \rangle e^{ik\cdot x} = \frac{i}{k^2 - m^2 + i\epsilon} + \cdots}$$

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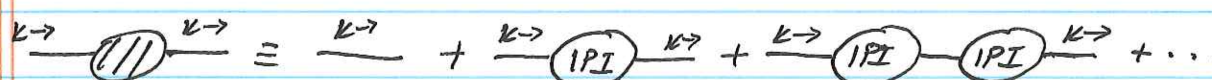
Now let's see what these renormalization conditions have to do with Feynman diagrams:

Define <u>1-particle irreducible</u> diagrams as those diagrams for which cutting any one line does not split the diagram into disconnected parts.

1PI diagrams

Not 1PI.

Calculating corrections to the propagator due to interactions is equivalent to summing over all Feynman diagrams w/ two external lines:



Renormalized propagator $\equiv \tilde{D}(k^2)$

Sum over all 1PI diagrams w/ two external lines $\equiv -i\,\tilde{\Pi}(k^2)\left(\frac{i}{k^2-m^2+i\epsilon}\right)^2$

from 2 external Feynman propagators

The sum over 1PI contributions to the propagator defines the <u>self energy</u> for the scalar field $\tilde{\Pi}(k^2)$.

Note that the sum over diagrams form a geometric series in powers of the self energy:

$$\tilde{D}(k^2) = \frac{i}{k^2-m^2+i\epsilon}\left[1 + \frac{\tilde{\Pi}(k^2)}{k^2-m^2+i\epsilon} + \left(\frac{\tilde{\Pi}(k^2)}{k^2-m^2+i\epsilon}\right)^2 + \dots\right]$$

$$= \frac{i}{k^2-m^2+i\epsilon}\,\frac{1}{1-\tilde{\Pi}(k^2)/(k^2-m^2+i\epsilon)}$$

$$\boxed{\tilde{D}(k^2) = \frac{i}{k^2-m^2-\tilde{\Pi}(k^2)+i\epsilon}}$$

$\tilde{\Pi}(k^2)$ is like a momentum-dependent mass.

The renormalization conditions can be expressed as conditions on $\tilde{\Pi}(k^2)$:

$\tilde{D}(k^2)$ has a pole at the physical mass $m^2 \Leftrightarrow \boxed{\tilde{\Pi}(m^2) = 0.}$

Residue of pole is $i \Leftrightarrow \boxed{\left.\frac{d\tilde{\Pi}}{dk^2}\right|_{k^2=m^2} = 0}$

To see the last relation, expand $\tilde{\Pi}(k^2)$ about $k^2=m^2$:

$$\tilde{\Pi}(k^2) = \underbrace{\tilde{\Pi}(m^2)}_{0} + \underbrace{\left.\frac{d\tilde{\Pi}(k^2)}{dk^2}\right|_{k^2=m^2}(k^2-m^2)}_{\text{changes residue of the pole unless } \left.\frac{d\tilde{\Pi}}{dk^2}\right|_{k^2=m^2}=0.} + \dots$$