Physics 722, Spring 2021
Problem Set 9
Due Thursday, April 29.

1. *Potential Energy from the Functional Integral*

Consider a real scalar field $\phi(x)$ coupled to a static background source $\rho(\mathbf{x})$, with action

$$S[\rho(\mathbf{x})] = \int d^4x \left[\frac{1}{2}\left(\partial_\mu \phi\right)^2 - \frac{m^2}{2}\phi^2 - \rho(\mathbf{x})\phi(x)\right].$$

It follows from our discussion of the functional integral that

$$\begin{aligned} \lim_{T\to\infty(1-i\epsilon)} e^{-i\Delta E_0 T} \quad &\equiv \quad \lim_{T\to\infty(1-i\epsilon)} \frac{\langle 0|\exp[-iH[\rho(\mathbf{x})]T]|0\rangle}{\langle 0|\exp[-iH[0]T]|0\rangle} \\ &= \quad \frac{\int \mathcal{D}\phi\ \exp\left[iS[\rho(\mathbf{x})]\right]}{\int \mathcal{D}\phi\ \exp\left[iS[0]\right]}. \end{aligned}$$

From this relation, calculate the the ground state energy in the theory with source $\rho(\mathbf{x})$, relative to that of the source-free theory, which should take the form

$$\Delta E_0 = \frac{1}{2}\int d^3x\, d^3x'\, \rho(\mathbf{x}')V(\mathbf{x}' - \mathbf{x})\rho(\mathbf{x}).$$

You should evaluate $V(\mathbf{x}' - \mathbf{x})$. Don't just leave it in the form of a Fourier transform.

Hint: Think about Gaussian integrals with a linear term in addition to the quadratic term in the argument of the exponential. Find a transformation from $\phi(\mathbf{x})$ to some $\tilde{\phi}(\mathbf{x})$ that would "complete the square" in the functional integral.

2. *Lie Groups*

a) Show that if the generators of a Lie group satisfy the Lie algebra,

$$\left[T^a, T^b\right] = i\, f^{abc}\, T^c,$$

then the generators in the adjoint representation, defined by

$$(T^a)_{bc} = -i\, f^{abc},$$

satisfy the Lie algebra.

b) Show that $T^2 = T^a T^a$ commutes with all the group generators. As a consequence, $(T^2)_{ij} = C_2(r)\,\delta_{ij}$, where the constant $C_2(r)$ is called the *quadratic Casimir* of the representation r.

c) Suppose the generators are normalized so that

$$\mathrm{Tr}\, T^a T^b = C(r)\,\delta^{ab},$$

where the constant $C(r)$ depends on the representation r. Suppose the generators in the representation r are $d(r) \times d(r)$ matrices. Show that

$$d(r) C_2(r) = d(G) C(r),$$

where G stands for the adjoint representation.

d) Suppose the generators of SU(N) in the fundamental representation are normalized by

$$\mathrm{Tr}\, T^a T^b = \frac{1}{2}\delta^{ab}.$$

Calculate the quadratic Casimir C_2 in this representation.

3. *Coulomb gauge*

Coulomb gauge is defined by the condition $\nabla \cdot \mathbf{A} = 0$. The Fadeev-Popov determinant is a constant in this case, as in the covariant gauges discussed in class. The determinant can then be absorbed in the normalization of the functional integral.

a) Show that the spatial components of the photon propagator are,

$$\frac{-i}{k^2 + i\epsilon}\left(g_{ij} + \frac{k_i k_j}{\mathbf{k}^2}\right).$$

b) Show that the 0-0 component of the photon propagator is,

$$\frac{i}{\mathbf{k}^2}.$$

c) What are the time-space components of the photon propagator?