Physics 722, Spring 2021
Problem Set 2, due Thursday, Feb 18.

1. *Derivative Interactions*

In class we mentioned the subtleties in dealing with derivative interactions. Here you will study a simple example in which you will observe that the naive handling of derivatives gives the correct answer.

Consider a free real scalar field with Lagrangian,

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi)^2 - \frac{m^2}{2}\phi^2.$$

We can equivalently write this as a Lagrangian in terms of a rescaled field $\tilde{\phi} \equiv Z^{-1/2}\phi$ as,

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \tilde{\phi})^2 - \frac{m^2}{2}\tilde{\phi}^2 + (Z-1)\left[\frac{1}{2}(\partial_\mu \tilde{\phi})^2 - \frac{m^2}{2}\tilde{\phi}^2\right].$$

The Fourier-transformed two-point function for the field ϕ is the usual free-field propagator:

$$\int d^4x\, e^{ik\cdot x}\langle 0|T\left(\phi(x)\phi(0)\right)|0\rangle = \frac{i}{k^2 - m^2 + i\epsilon}.$$

a) Given the relationship between $\tilde{\phi}$ and ϕ, the propagator for the field $\tilde{\phi}$ is a simple rescaling of the propagator for the field ϕ. What is

$$\int d^4x\, e^{ik\cdot x}\langle 0|T\left(\tilde{\phi}(x)\tilde{\phi}(0)\right)|0\rangle?$$

In the remainder of this problem you will reproduce the result of part (a) by evaluating $\langle 0|T\left(\tilde{\phi}(x)\tilde{\phi}(0)\right)|0\rangle$ in perturbation theory, where you are to think of the the terms in the Lagrangian proportional to $(Z-1)$ as being interaction terms.

b) What are the Feynman rules for the two-point vertices corresponding to the interactions in this theory, treating the derivatives in the interactions naively?

c) By summing over *all* diagrams that contribute to the Fourier transformed two-point function, $\int d^4x\, e^{ik\cdot x}\langle 0|T\left(\tilde{\phi}(x)\tilde{\phi}(0)\right)|0\rangle$, show that you recover the result of part (a).

2. *Scalar Self Energy*

Consider the theory of a real scalar field coupled to a Dirac spinor field,

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi)^2 - \frac{\mu^2}{2}\phi^2 + \overline{\psi}(i\partial\!\!\!/ - m)\psi - g\,\overline{\psi}\psi\phi - \frac{g_3}{3!}\,\phi^3 - \frac{\lambda}{4!}\,\phi^4 + \text{counterterms}.$$

Calculate the one-loop renormalized self energy $\widetilde{\Pi}(k^2)$ for the scalar field ϕ. $\widetilde{\Pi}(k^2)$ should satisfy the renormalization conditions $\widetilde{\Pi}(\mu^2) = 0$ and $d\widetilde{\Pi}/dk^2|_{k^2=\mu^2} = 0$. Your result should be left in terms of integrals over a single Feynman parameter.

3. *Volume of a d-dimensional sphere*

We used the volume of a 3-dimensional unit sphere in the evaluation of integrals over Euclidean momentum. You will derive the result here.

a) Compute the $(d+1)$-dimensional integral

$$I_{d+1} \equiv \int d^{d+1}x\, e^{-(x_1^2+x_2^2+\cdots+x_{d+1}^2)}.$$

b) Write the integral in spherical coordinates and separate out the angular part of the integral. By a suitable change of variables, use the definition of the gamma function,

$$\int_0 x^{t-1} e^{-x}\, dx,$$

to evaluate the integral in terms of the gamma function and the volume of a d-dimensional unit sphere.

c) By comparing the results of parts (a) and (b), determine the volume of the d-dimensional unit sphere.