Physics 722, Spring 2021
Problem Set 10
Due Thursday, May 6.

Your final exam will be a take-home exam. The exam will be available Saturday, May 8 and will be due Wednesday, May 12 at 5pm. The exam should be completed during a contiguous 72-hour period within that window, and you should not spend more than 12 hours working on the exam.

1. *Current Algebra*

We have seen a number of examples of symmetries generated by the corresponding conserved charges. For example, translations are generated by the momentum operator, and rotations are generated by the angular momentum. Those charges satisfy the algebra of the corresponding symmetry group. For example, the angular momentum operators satisfy the SO(3) algebra:

$$[L_j, L_k] = i\,\epsilon_{jkl}\,L_l.$$

It is generally true that conserved charges satisfy the corresponding symmetry algebra. Here you will study a generic theory of scalar fields with a continuous global symmetry.

Consider the Lagrangian,

$$\mathcal{L} = \sum_{i=1}^{N} \partial_\mu \phi_i^\dagger\, \partial^\mu \phi_i - V(\phi_i),$$

where $V(\phi_i)$ is a Hermitian function of ϕ_i and $\phi_i^\dagger$ without spacetime derivatives.

Suppose $\mathcal{L}$ has a group of symmetries parametrized by $\dim G$ parameters θ^a,

$$\phi_j \to \phi_j + i\,\theta^a\,T^a_{jk}\,\phi_k + \mathcal{O}(\theta^2),$$

generated by the $N \times N$ matrices T^a, $a = 1, \ldots, \dim G$, satisfying the algebra,

$$\left[T^a, T^b\right] = i\,f^{abc}\,T^c.$$

a) What are the conserved currents due to the symmetries (by Noether's theorem)? What are the corresponding charges?

b) Using the equal-time commutation relations,

$$[\pi_i(\mathbf{x},t), \phi_j(\mathbf{y},t)] = -i\,\delta_{ij}\,\delta^3(\mathbf{x}-\mathbf{y}),$$

show that the currents satisfy the *current algebra*,

$$\left[J_0^a(\mathbf{x},t), J_0^b(\mathbf{y},t)\right] = i\,f^{abc}\,J_0^c(\mathbf{x},t)\,\delta^3(\mathbf{x}-\mathbf{y}).$$

c) Show that $\left[Q^a, J_0^b(\mathbf{x},t)\right] = i\,f^{abc}\,J_0^c(\mathbf{x},t)$.

d) Show that the charges satisfy the *charge algebra*,

$$\left[Q^a, Q^b\right] = i\,f^{abc}\,Q^c.$$

Comments:
In your free time you can check that the same relations hold in a theory of fermions, or in a theory with both fermions and scalars. The current algebra depends only on the symmetry group, not on the details of the underlying theory.

Current algebra provides a powerful tool for deriving selection rules and relating the scattering amplitudes of different processes related by symmetry transformations.

2. *Pion-Nucleon Interactions*

The pions are pseudoscalar mesons that form a triplet under global SU(2) isospin. The proton and neutron form an SU(2) doublet of Dirac ferions. In this problem you will consider the Yukawa interactions between pions and nucleons.

Define the nucleon doublet,

$$N = \begin{pmatrix} p \\ n \end{pmatrix},$$

which transforms under an SU(2) transformation by $N \to g(\theta^a)N$, where $g(\theta^a) = \exp[i\theta^a\sigma^a/2]$ and σ^a, $a = 1,2,3$ are the Pauli sigma matrices.

The pions form a triplet π^a, $a = 1,2,3$, and we define,

$$\pi = \pi^a\,\frac{\sigma^a}{2},$$

which transforms as $\pi \to g\pi g^{-1}$.

Consider the theory described by the Lagrangian,

$$\mathcal{L} = \overline{N}(i\partial\!\!\!/ - m)N + \mathrm{Tr}\,(\partial_\mu \pi)(\partial^\mu \pi) - ig\,\overline{N}\gamma^5 \pi N.$$

a) Show that $\mathcal{L}$ is invariant under SU(2) isospin transformations.

b) Expand the Lagrangian in components, *i.e.*

$$\mathcal{L} = \overline{p}(i\partial\!\!\!/ - m)p + \cdots - \frac{ig}{2}\,\overline{p}\gamma^5 p\pi^3 + \cdots$$

c) Define $\pi^0 = \pi^3$ and $\pi^\pm = (\pi^1 \mp i\pi^2)/\sqrt{2}$. Write $\mathcal{L}$ in terms of p, n, π^0 and $\pi^\pm$.

d) Calculate the 1PI pion self energy diagrams $\Pi^{ab}(k^2)$ contributing to the Fourier transform of $\langle 0|T(\pi^a(x)\pi^b(0))|0\rangle$. You do not need to do the momentum integrals. You should evaluate all group theoretic factors, and leave your result in terms of the 1-loop scalar self energy diagram $\Pi(k^2)$ for a single real scalar ϕ and a single Dirac fermion ψ, Yukawa coupled with Lagrangian,

$$\mathcal{L}_1 = \overline{\psi}(i\partial\!\!\!/ - m)\psi + \frac{1}{2}(\partial_\mu \phi)^2 - ig\overline{\psi}\gamma^5\psi\phi.$$