The Higgs Mechanism: Spontaneously broken gauge invariance.
(Cheng+Li ch. 8.3)

Unbroken global symmetries lead to degeneracies between states, selection rules and relations between scattering amplitudes.

Broken continuous global symmetries lead to massless particles.

Broken gauge invariance leads to massive vector fields. This is the Higgs mechanism, and in the Standard Model is responsible for the masses of the W and Z bosons.

Abelian Example: $\mathcal{L} = (D_\mu \phi)^\dagger (D^\mu \phi) + \mu^2 \phi^\dagger \phi - \lambda(\phi^\dagger\phi)^2 - \frac{1}{4} F_{\mu\nu} F^{\mu\nu}$

$D_\mu \phi = (\partial_\mu - i g A_\mu)\phi$

$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$

Gauge invariance: $\phi(x) \to e^{-i\alpha(x)} \phi(x)$

$A_\mu(x) \to A_\mu(x) - \frac{1}{g} \partial_\mu \alpha(x)$

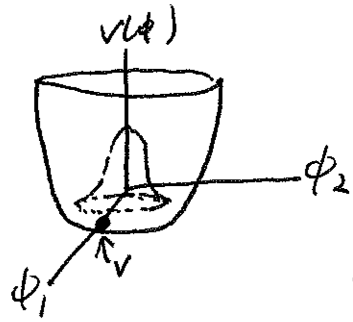
$\mu^2 > 0$: Minimum of potential $V(\phi) = -\mu^2 \phi^\dagger \phi + \lambda(\phi^\dagger\phi)^2$ is at $|\phi| = v/\sqrt{2}$, $v = (\mu^2/\lambda)^{1/2}$.

Hence, $|\langle 0|\phi(x)|0\rangle| = v/\sqrt{2}$.

The vacuum expectation value (VEV) of $\phi(x)$ breaks the gauge invariance because under a gauge transformation

$\langle 0|\phi(x)|0\rangle \to \langle 0|\phi(x)|0\rangle e^{-i\alpha(x)}$

We can choose $\langle 0|\phi(x)|0\rangle$ to be real, $\langle 0|\phi(x)|0\rangle = v/\sqrt{2}$.



Define $\phi(x) = \frac{1}{\sqrt{2}}\left(\phi_1(x) + i\,\phi_2(x)\right)$,
ϕ_1 and ϕ_2 real.

Define $\phi_1' = \phi_1 - v$
$\phi_2' = \phi_2$

The potential is $V(\phi) = -\mu^2 \phi^\dagger\phi + \lambda(\phi^\dagger\phi)^2$

$$= -\frac{\mu^2}{2}\left(\phi_1^2+\phi_2^2\right) + \frac{\lambda}{4}\left(\phi_1^2+\phi_2^2\right)^2$$

$$= -\frac{\mu^2}{2}\left((\phi_1'+v)^2 + \phi_2'^2\right) + \frac{\lambda}{4}\left((\phi_1'+v)^2+\phi_2'^2\right)^2$$

$$= -\frac{\mu^4}{4\lambda} + \mu^2\phi_1'^2 + \lambda v\,\phi_1'\left(\phi_1'^2+\phi_2'^2\right) + \frac{\lambda}{4}\left(\phi_1'^2+\phi_2'^2\right)^2$$

Note that ϕ_1' has mass $\sqrt{2}\mu$, but ϕ_2' would be massless. If this were the end of the story, ϕ_2' would be the Goldstone boson of the spontaneously broken $U(1)$. However, $\phi_2'(x)$ can be transformed away by a gauge transformation, so we have to work harder to identify the physical degrees of freedom.

Consider the kinetic term for ϕ:

$$|D_\mu\phi|^2 = |(\partial_\mu - igA_\mu)\phi|^2$$

$$= \frac{1}{2}\left[(\partial_\mu - igA_\mu)(v+\phi_1'+i\phi_2')\right]\left[(\partial_\mu + igA_\mu)(v+\phi_1'-i\phi_2')\right]$$

$$= \frac{1}{2}\left(\partial_\mu\phi_1' + gA_\mu\phi_2'\right)^2 + \frac{1}{2}\left(\partial_\mu\phi_2' - gA_\mu\phi_1'\right)^2$$
$$- gvA^\mu\left(\partial_\mu\phi_2' + gA_\mu\phi_1'\right) + \underbrace{\frac{g^2v^2}{2}A_\mu A^\mu}_{\text{Mass term for } A^\mu \text{ !!!}}$$

The term $gvA^\mu \partial_\mu \phi_2'$ mixes A^μ with the would-be Goldstone boson ϕ_2'. To make the physical degrees of freedom clear we gauge this term away.

Define $\phi(x) = \frac{1}{\sqrt{2}}(v + \eta(x)) \exp(i\xi(x)/v)$

$$= \frac{1}{\sqrt{2}}\Big(v + \underset{\text{like } \phi_1'(x)}{\eta(x)} + i\,\underset{\text{like } \phi_2'(x)}{\xi(x)} + \ldots\Big)$$

Define the gauge-transformed fields (Unitary Gauge)

$$\phi^u(x) \equiv \exp(-i\xi(x)/v)\,\phi(x) = \frac{1}{\sqrt{2}}(v + \eta(x))$$
$$B_\mu(x) \equiv A_\mu(x) - \frac{1}{gv}\partial_\mu \xi(x)$$

The covariant derivative $D_\mu \phi$ transforms like ϕ, so

$$D_\mu \phi = \exp(i\xi(x)/v)(\partial_\mu \phi^u - igB_\mu \phi^u)$$

$$= \exp(i\xi(x)/v)(\partial_\mu \eta(x) - igB_\mu(v + \eta(x)))/\sqrt{2}$$

$$|D_\mu \phi|^2 = \frac{1}{2}|\partial_\mu \eta - igB_\mu(v+\eta)|^2$$

Also, $F_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu$.

The Lagrangian is $\mathcal{L} = \frac{1}{2}|\partial_\mu \eta - igB_\mu(v+\eta)|^2 + \frac{\mu^2}{2}(v+\eta)^2$
$$- \frac{\lambda}{4}(v+\eta)^4 - \frac{1}{4}(\partial_\mu B_\nu - \partial_\nu B_\mu)^2$$

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \eta)^2 - \mu^2\eta^2 - \frac{1}{4}(\partial_\mu B_\nu - \partial_\nu B_\mu)^2 + \frac{1}{2}(gv)^2 B_\mu B^\mu$$
$$+ \frac{1}{2} g^2 B_\mu B^\mu \eta(2v+\eta) - \lambda v^2 \eta^3 - \frac{1}{4}\lambda\eta^4$$

Note that the field $\xi(x)$ has completely disappeared from the Lagrangian.
However, the field B_μ has mass gv.

It is said that the gauge field has eaten the would-be Goldstone boson to become massive.

Note that the number of propagating degrees of freedom has not changed. A massless vector field has 2 propagating degrees of freedom (helicity ± 1).
A massive vector field has an additional helicity 0 ("longitudinal") mode.

Quantization of Spontaneously Broken Gauge Theories

In unitary gauge the degrees of freedom are manifest. The free part of the Lagrangian describing the massive components of the gauge field has the Proca form:

$$\mathcal{L}_0 = -\frac{1}{4}(\partial_\mu A_\nu^a - \partial_\nu A_\mu^a)(\partial^\mu A^{\nu a} - \partial^\nu A^{\mu a}) + \frac{m^2_{ab}}{2} A_\mu^a A^{\mu b}$$

The mass matrix m^2_{ab} is real and symmetric, so it can be diagonalized with an orthogonal matrix. In the diagonalized basis, $\mathcal{L}_0 = -\frac{1}{4}(\partial_\mu A_\nu^a - \partial_\nu A_\mu^a)^2 + \sum_a \frac{m_a^2}{2} A_\mu^a A^{\mu a}$.

The propagator for the massive gauge fields have the form $\langle 0|T A_\mu^a(x) A_\nu^b(y)|0\rangle$

$$= \frac{\int \mathcal{D}A^\alpha \exp(i\int d^4x \mathcal{L}_0)\, A_\mu^a(x) A_\nu^b(y)}{\int \mathcal{D}A^\beta \exp(i\int d^4x \mathcal{L}_0)}$$

(Exercise) $$= \int \frac{d^4k}{(2\pi)^4} \frac{-i e^{-ik\cdot(x-y)}}{k^2 - m_a^2 + i\epsilon}\left(g_{\mu\nu} - \frac{k_\mu k_\nu}{m_a^2}\right)\delta^{ab}$$

Hence, the Feynman rule for the propagator in unitary gauge is

$$\boxed{\overset{\mu}{\underset{a}{}} \overset{k\rightarrow}{\sim\!\sim\!\sim} \overset{\nu}{\underset{b}{}} \quad \frac{-i}{k^2 - m_a^2 + i\epsilon}\left(g_{\mu\nu} - \frac{k_\mu k_\nu}{m_a^2}\right)\delta^{ab}}$$

This form of the propagator would seem to make integrals over k^μ more divergent than for the scalar field, because of the $k_\mu k_\nu$ term. This puts renormalizability of the theory in greater question. However, there is an average over gauges that makes the large-k^μ behavior more well-behaved.

R_ξ gauges

Consider the Abelian Higgs model:

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + (D_\mu \phi)^\dagger D^\mu \phi + \mu^2 \phi^\dagger \phi - \lambda (\phi^\dagger \phi)^2$$

Expanding about the VEV $\langle \phi \rangle = v/\sqrt{2} = \frac{1}{\sqrt{2}} \left(\frac{\mu^2}{\lambda}\right)^{1/2}$,

$\mathcal{L} \supset + g v A^\mu \partial_\mu \phi_2'$, where $\phi = \frac{1}{\sqrt{2}}(v + \phi_1' + i\phi_2')$.

We can eliminate the term mixing the gauge field and the would-be Goldstone boson by averaging over gauges w/ gauge-fixing condition

$$\boxed{G[A_\mu, \phi] = \partial^\mu A_\mu + \xi m_A \phi_2 - f(x) = 0},$$

$m_A = gv$.

Temporarily disregarding the Fadeev-Popov determinant (which gives rise to ghosts), we insert into the functional integral:

$$\int \mathcal{D}f(x) \exp\left(-\frac{i}{2\xi} \int f(x)^2 d^4x\right) \delta\left(G[A_\mu, \phi]\right)$$

$$= \exp\left(\frac{-i}{2\xi} \int d^4x \left(\partial^\mu A_\mu + \xi m_A \phi_2\right)^2\right) \equiv \exp\left(i \int d^4x\, \mathcal{L}_{g.f.}\right)$$

where the gauge-fixing Lagrangian has the form

$$\boxed{\mathcal{L}_{g.f.} = -\frac{1}{2\xi}\left(\partial^\mu A_\mu + \xi m_A \phi_2\right)^2} \leftarrow R_\xi \text{ gauge}$$

The free part of the Lagrangian, including $\mathcal{L}_{g.f.}$, is

$$\boxed{\begin{aligned}\mathcal{L}_0 = \frac{1}{2}\left((\partial^\mu \phi_1')^2 - 2\mu^2 \phi_1'^2\right) + \frac{1}{2}\left((\partial^\mu \phi_2')^2 - \xi m_A^2 \phi_2'^2\right) \\ - \frac{1}{4}(\partial_\mu A_\nu - \partial_\nu A_\mu)^2 + \frac{1}{2} m_A^2 A_\mu A^\mu - \frac{1}{2\xi}(\partial^\mu A_\mu)^2\end{aligned}}$$

Note that the mixing between A_μ and ϕ_2' has been eliminated, but unlike in unitary gauge the field ϕ_2' propagates.

Consider the gauge field part of the action:

$$S = \int d^4x \, \frac{1}{2} A_\mu \left(g^{\mu\nu}\partial^2 - \partial^\mu\partial^\nu + m_A^2 g^{\mu\nu} + \frac{1}{\xi}\partial^\mu\partial^\nu \right) A_\nu$$

$$= \int d^4x \, \frac{1}{2} A_\mu \left[(\partial^2 + m_A^2)\left(g^{\mu\nu} - \frac{\partial^\mu\partial^\nu}{\partial^2}\right) + \left(\frac{1}{\xi}\partial^2 + m_A^2\right)\left(\frac{\partial^\mu\partial^\nu}{\partial^2}\right)\right] A_\nu$$

The Kernel of the Gaussian functional integral is

$$K^{\mu\nu}(x, x') = -i\,\delta^4(x-x')\left[(\partial^2 + m_A^2)\left(g^{\mu\nu} - \frac{\partial^\mu\partial^\nu}{\partial^2}\right) + \left(\frac{1}{\xi}\partial^2 + m_A^2\right)\left(\frac{\partial^\mu\partial^\nu}{\partial^2}\right)\right]$$

We can invert the transverse and longitudinal parts separately:

$$K^{-1}_{\nu\alpha}(x', x'') = \int \frac{d^4k}{(2\pi)^4}\left[\frac{-i}{k^2 - m_A^2 + i\epsilon}(P_T)_{\nu\alpha} - \frac{i\,\xi}{k^2 - \xi m_A^2 + i\epsilon}(P_L)_{\nu\alpha}\right] e^{-ik\cdot(x'-x'')}$$

where $(P_T)_{\nu\alpha} = g_{\nu\alpha} - \frac{k_\nu k_\alpha}{k^2}$, $(P_L)_{\nu\alpha} = \frac{k_\nu k_\alpha}{k^2}$.

$$K^{-1}_{\nu\alpha}(x', x'') = \int \frac{d^4k}{(2\pi)^4} \left[\frac{-i}{k^2 - m_A^2 + i\epsilon} \left(g_{\nu\alpha} - \frac{k_\nu k_\alpha}{k^2} \right) - \frac{i\xi}{k^2 - \xi m_A^2 + i\epsilon} \frac{k_\nu k_\alpha}{k^2} \right] e^{-ik\cdot(x'-x'')}$$

$$= \int \frac{d^4k}{(2\pi)^4} \left[\frac{-i}{k^2 - m_A^2 + i\epsilon} \left(g_{\nu\alpha} - \frac{k_\nu k_\alpha}{k^2} \left(1 - \frac{\xi(k^2 - m_A^2)}{k^2 - \xi m_A^2 + i\epsilon} \right) \right) \right]$$

$$= \int \frac{d^4k}{(2\pi)^4} \left[\frac{-i}{k^2 - m_A^2 + i\epsilon} \left(g_{\nu\alpha} - (1-\xi) \frac{k_\nu k_\alpha}{k^2 - \xi m_A^2 + i\epsilon} \right) \right]$$

<u>Exercise</u>: Check $\int d^4x' \, K^{\mu\nu}(x, x') K^{-1}_{\nu\alpha}(x', x'') = \delta^4(x - x'')\delta^\mu_\alpha$

From $K^{-1}_{\nu\alpha}$ we read off the Feynman rule for the massive gauge boson in R_ξ gauge:

$$\boxed{\mu \sim\sim\sim \nu \qquad \frac{-i}{k^2 - m_A^2 + i\epsilon} \left(g_{\nu\alpha} - (1-\xi) \frac{k_\nu k_\alpha}{k^2 - \xi m_A^2 + i\epsilon} \right)}$$

☆ Note that for large k_ν, the propagator scales like the scalar field propagator for any finite ξ.

In R_ξ gauge the would-be Goldstone boson also propagates:

$$\mathcal{L}_0 \supset \frac{1}{2}\left((\partial_\mu \phi_2')^2 - \xi m_A^2 \phi_2'^2 \right)$$

$$\boxed{\phi_2' \xrightarrow{k} \phi_2' \qquad \frac{i}{k^2 - \xi m_A^2 + i\epsilon}}$$

☆ As $\xi \to \infty$, ϕ_2' decouples and the gauge field propagator approaches the unitary gauge propagator.

Summary:

The Feynman rules for the propagators are:

$k \rightarrow$ ϕ_1' $\quad \dfrac{i}{k^2 - 2\mu^2 + i\epsilon}$

$k \rightarrow$ ϕ_2' $\quad \dfrac{i}{k^2 - \xi m^2 + i\epsilon}$

$\mu \quad \nu \quad \dfrac{-i}{k^2 - M^2 + i\epsilon}\left[g^{\mu\nu} - (1-\xi)\dfrac{k^\mu k^\nu}{k^2 - \xi m^2}\right]$

$$= -i\left[\frac{g^{\mu\nu} - k^\mu k^\nu/m^2}{k^2 - m^2 + i\epsilon} + \frac{k^\mu k^\nu/m^2}{k^2 - \xi m^2 + i\epsilon}\right]$$

In the R_ξ gauge the would-be Goldstone boson ϕ_2' propagates, but the propagator depends on the gauge fixing parameter ξ.

But now the vector boson propagator falls like $\frac{1}{k^2}$ at large momentum, so the high energy behavior is manifestly more mild than in unitary gauge.

(Fujikawa, Lee, Sanda (1972); Yao (1973))

In QED in covariant gauges the Fadeev-Popov determinant is independent of the fields and can be factored out of the functional integral. However, in R_ξ gauge it's not so simple.

The gauge fixing condition is

$$G(A_\mu, \phi) = \partial_\mu A^\mu + \xi m \phi_2 - f(x) = 0$$

In terms of the shifted fields $\phi_1' = \phi_1 - v$, $\phi_2' = \phi_2$, the gauge invariance is (infinitessimally)

$$\phi_1' \to \phi_1'(x) - \alpha(x)\,\phi_2'(x)$$
$$\phi_2' \to \phi_2' + \alpha(x)\,(v + \phi_1')$$
$$A_\mu \to A_\mu - \frac{1}{e}\partial_\mu \alpha(x)$$

The Fadeev-Popov determinant is

$$\det\left(\frac{\delta G}{\delta \alpha}\right) = \det\left(-\frac{1}{e}\partial_\mu\partial^\mu + \xi m (v + \phi_1')\right)$$

Introducing Fadeev-Popov ghost fields $c(x), \bar{c}(x)$, we can absorb the determinant into a functional integral over the ghosts, with ghost Lagrangian

$$\boxed{\mathcal{L}^{R_\xi}_{ghost} = \bar{c}\left(-\partial_\mu\partial^\mu + \xi m^2\left(1 + \frac{\phi_1'}{v}\right)\right)c}$$

ghost coupling to physical scalar.

The non-Abelian generalization of the Higgs mechanism is straightforward, as is the discussion of unitary gauge and R_ξ-gauge.
(See Peskin + Schroeder Ch. 20.1 and 21.1,
Cheng + Li Ch. 8.3, 9.2)

In the Standard Model, the <u>Higgs doublet</u> is an $SU(2)_W$ doublet and is charged under $U(1)_Y$.

weak gauge invariance

Hypercharge gauge invariance

The Higgs field acquires a vacuum expectation value, which spontaneously breaks $SU(2)_W \times U(1)_Y$ to $U(1)_{EM}$.

Electromagnetism

That is why, if the Standard Model is correct, the $W^\pm$ and Z bosons are heavy, while the photon is massless.