Quantization of the Non-Abelian Gauge Field

As for QED, we impose a gauge-fixing condition using Fadeev & Popov's trick:
To impose the condition $G(A_\mu^a(x)) = 0$, use

$$1 = \int \mathcal{D}\alpha^a(x)\ \delta\left(G(A_\mu^{a(\alpha)})\right) \det\left(\frac{\delta G(A_\mu^{a(\alpha)})}{\delta\alpha}\right)$$

The infinitessimal gauge transformation is:

$$A_\mu^{a(\alpha)} = A_\mu^a + \frac{1}{\tilde{e}}\partial_\mu\alpha^a + f^{abc}A_\mu^b\alpha^c$$

$\tilde{e}$ ← gauge coupling

$$= A_\mu^a + \frac{1}{\tilde{e}}D_\mu\alpha^a$$

D_μ ← covariant derivative on adjoint rep.

As in QED:
1) Under the gauge transformation, $\mathcal{D}A^{(\alpha)} = \mathcal{D}A$.
2) The integral $\int\mathcal{D}\alpha^a$ can be factored out because everything in the functional integral is gauge invariant.

Write $\int\mathcal{D}A\ \mathcal{O}(A)e^{iS[A]}$

$$= \int\mathcal{D}\alpha\int\mathcal{D}A\ e^{iS[A]}\delta(G(A))\,\mathcal{O}(A)\det\left(\frac{\delta G(A^{(\alpha)})}{\delta\alpha}\right)$$

As in QED, we can average over generalized Lorenz gauges, $G(A) = \partial_\mu A^{\mu a} - f^a(x)$, for some functions $f^a(x)$.

The main difference from the analogous treatment of QED is that now we can't factor out the Fadeev-Popov determinant.

From the infinitessimal form of the gauge transformation, the FP determinant is

$$\boxed{\det\left(\frac{\delta G(A^{(\alpha)})}{\delta \alpha}\right) = \det\left(\frac{1}{e}\partial^\mu D_\mu\right)}$$

Fadeev & Popov's trick for handling the determinant is to introduce a fictitious set of interactions between the gauge fields and fictitious <u>fermionic scalar fields</u> called <u>ghosts</u>, such that the functional integral over the ghosts reproduces the FP determinant.

We will see how this works shortly.

The derivation of the propagator mimics that in QED, with the result:

$$\boxed{D^{ab}_{\mu\nu}(x,y) = \int \frac{d^4k}{(2\pi)^4} \frac{-i}{k^2+i\epsilon}\left(g_{\mu\nu} - (1-\xi)\frac{k_\mu k_\nu}{k^2}\right)\delta^{ab} e^{-ik\cdot(x-y)}}$$

Fadeev-Popov Ghosts

We have derived an expression for correlators of gauge invariant operators in Yang-Mills Theory in terms of the functional integral:

$$\boxed{\langle 0|T(\mathcal{O}(A_\mu))|0\rangle = \frac{\int \mathcal{D}A\ \mathcal{O}(A)\ \exp\left[i\int_{-T}^{T} d^4x\left(\mathcal{L} - \frac{1}{2\xi}(\partial_\mu A^{\mu a})^2\right)\right]\det\left(\frac{\delta G(A^{(\alpha)})}{\delta\alpha}\right)}{\int \mathcal{D}A\ \exp\left[i\int_{-T}^{T} d^4x\left(\mathcal{L} - \frac{1}{2\xi}(\partial_\mu A^{\mu a})^2\right)\right]\det\left(\frac{\delta G(A^{(\alpha)})}{\delta\alpha}\right)}}$$

where $\boxed{\det\left(\frac{\delta G(A^{(\alpha)})}{\delta\alpha}\right) = \det\left(\frac{1}{e}\partial^\mu D_\mu\right)}$

covariant deriv. in adjoint rep.

We also know that determinants arise from Gaussian functional integrals over fermions:

$$\int \mathcal{D}\bar{\Psi}\mathcal{D}\Psi\ e^{i\int d^4x\, d^4y\, \bar{\Psi}(x) K(x,y)\Psi(y)} = \det K$$

Hence, if we introduce a set of (fictitious) <u>fermionic scalar fields</u> $c^a(x)$, $\bar{c}^a(x)$ transforming in the adjoint representation of the gauge group, we can write:

$$\det\left(\frac{1}{e}\partial^\mu D_\mu\right) = \int \mathcal{D}c\,\mathcal{D}\bar{c}\ \exp\left[i\int d^4x\ \bar{c}^a\left(-\partial^\mu D_\mu^{ab}\right)c^b\right]$$

It is as if we have added to the Lagrangian a term,

$$\boxed{\mathcal{L}_{ghost} = \bar{c}^a(-\partial^\mu D_\mu^{ab})c^b}$$

(Factor of $\frac{1}{\tilde{e}}$ absorbed in def. of c^a)

and integrated over the ghost fields c^a, $\bar{c}^a$

The new fields c^a and $\bar{c}^a$ are not physical, and can not appear as external fields. They are just a trick to evaluate the functional determinant.

The strange thing about c^a and $\bar{c}^a$ is that they have no Lorentz indices, so they are Lorentz scalars, but they are anticommuting. Ghosts violate the usual relation between spin and statistics. Ghosts are necessary to restore the unitarity of S-matrix elements, which would be violated using the usual Feynman rules w/o ghosts. Hence, it is sometimes said that the ghosts cancel the unphysical degrees of freedom in the gauge field.

The Lagrangian $\mathcal{L}_{ghost}$ gives rise to Feynman rules for the ghost field.

$$\mathcal{L}_{ghost} = \bar{c}^a(-\partial_\mu\partial^\mu\delta^{ac} - \tilde{e}\overleftarrow{\partial}^\mu A_\mu^b f^{abc})c^c$$

The first term in $\mathcal{L}_{ghost}$ gives the ghost propagator:

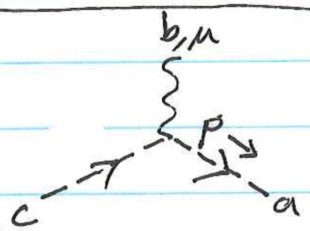
$$\langle c^a(x)\bar{c}^b(y)\rangle = \int \frac{d^4k}{(2\pi)^4} \frac{i}{k^2+i\epsilon} \delta^{ab} e^{-ik\cdot(x-y)}$$

$$\boxed{\frac{i\delta^{ab}}{p^2+i\epsilon}}$$

Ghost propagator

The arrow represents the flow of "ghost charge" from $\bar{c}$ to c.

The second term gives the gauge boson-ghost vertex



$$\boxed{\tilde{e} f^{abc} p^\mu}$$

Ghost vertex

In summary, the effective gauge-fixed Lagrangian for Yang-Mills Theory is given by

$$\boxed{\mathcal{L}_{YM}^{(\text{gauge fixed})} = -\frac{1}{4}F^a_{\mu\nu}F^{\mu\nu a} - \frac{1}{2\xi}(\partial_\mu A^{\mu a})^2 + \bar{c}^a(-\partial^\mu D_\mu^{ab})c^b}$$

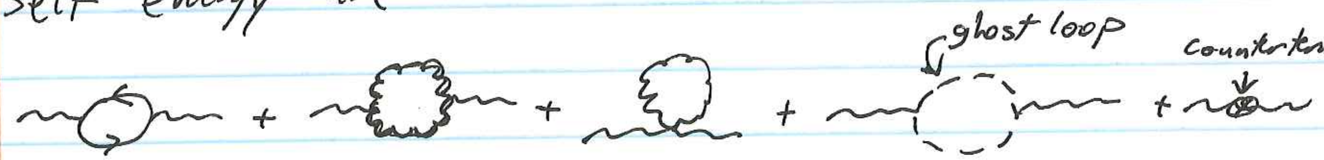
To include fermions, add $\mathcal{L}_{fermion}$,

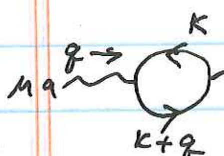
$$\boxed{\mathcal{L}_{fermion} = \bar{\Psi}(i\not{D}-m)\Psi}$$

Gauge Boson Self Energy

We are all set to begin calculating in Yang-Mills Theories. We'll do some sample calculations to get you started.

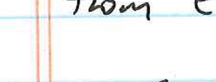
1-loop diagrams contributing to the gauge boson self energy are:



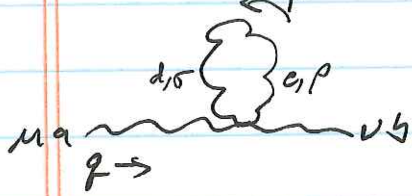


$$= (-i\tilde{e})^2 \underbrace{(-1)}_{\text{Fermion loop}} \int \frac{d^4k}{(2\pi)^4} \mathrm{Tr}\left[\gamma^\mu \frac{i}{\not{k} - m + i\epsilon} \gamma^\nu \frac{i}{\not{k} + \not{q} - m + i\epsilon}\right] \mathrm{Tr}(T^a T^b)$$

This diagram is equivalent to the analogous diagram in QED, except for the group theory factor $\mathrm{Tr}(T^a T^b)$ from the vertices, which simply factors out. Borrowing from expressions we derived for the photon self energy,



$$= -4i\tilde{e}^2 (g^{\mu\nu} q^2 - q^\mu q^\nu)\, \mathrm{Tr}(T^a T^b) \int_0^1 dx \frac{1}{(4\pi)^{d/2}} \Gamma(2 - d/2)\, 2x(1-x) \left(\frac{\overset{\text{Dim Reg renormalization scale}}{M^2}}{m^2 - x(1-x) q^2}\right)^{\frac{4-d}{2}}$$



Feynman gauge propagator

$$\frac{1}{2}\int\frac{d^4p}{(2\pi)^4}\,\frac{-i\,g_{\rho\sigma}}{p^2+i\epsilon}\,\delta^{cd}\left(-i\tilde{e}^2\right)$$

$$\Big[f^{abe}f^{cde}\left(g^{\mu\rho}g^{\nu\sigma}-g^{\mu\sigma}g^{\nu\rho}\right)$$
$$+f^{ace}f^{bde}\left(g^{\mu\nu}g^{\rho\sigma}-g^{\mu\sigma}g^{\nu\rho}\right)$$
$$+f^{ade}f^{bce}\left(g^{\mu\nu}g^{\rho\sigma}-g^{\mu\rho}g^{\nu\sigma}\right)\Big]$$

The first term in brackets vanishes when multiplied by δ^{cd} by the antisymmetry of f^{cde}.

The second two terms can be expressed in terms of the <u>quadratic Casimir</u> of the adjoint rep,

$$T^a_r\,T^a_r \equiv C_2(r)\,\mathbb{1}$$

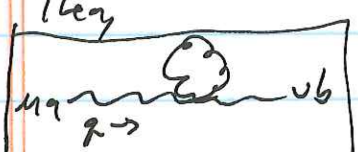
quadratic Casimir of rep r. C_2 commutes w/ all elements of the algebra.

In adjoint rep, $(T^b_{adj})_{ac} = i f^{abc}$

$$\left(T^b_{adj}T^b_{adj}\right)_{ad} = \left(if^{abc}\right)\left(if^{cbd}\right) = C_2(G)\,\delta_{ad}$$

$$\rightarrow f^{abc}f^{dbc} = C_2(G)\,\delta_{ad}$$

Then



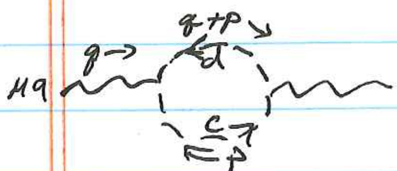
$$= \frac{1}{2}\int\frac{d^4p}{(2\pi)^4}\left(-\tilde{e}^2\right)\frac{1}{p^2+i\epsilon}$$

$$\Big[C_2(G)\,\delta^{ab}\left(g^{\mu\nu}\cdot 4-g^{\mu\nu}\right)$$
$$+C_2(G)\,\delta^{ab}\left(g^{\mu\nu}\cdot 4-g^{\mu\nu}\right)\Big]$$

$$\boxed{= -\tilde{e}^2\,C_2(G)\,\delta^{ab}\cdot 3g^{\mu\nu}\int\frac{d^4p}{(2\pi)^4}\,\frac{1}{p^2+i\epsilon}}$$

Using dim reg, $\int \frac{d^4p}{(2\pi)^4}\frac{1}{p^2+i\epsilon}$ has a pole at $d=2$,

but vanishes as $d \to 4$. Hence, the diagram  can be discarded.



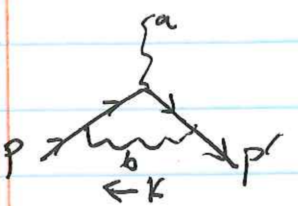
$$= (-1)\int \frac{d^4p}{(2\pi)^4}\frac{i}{p^2+i\epsilon}\frac{i}{(p+q)^2+i\epsilon} f^{dac}(p+q)^\mu f^{cbd} p^\nu \tilde{e}^2$$

ghost loop

$$= -C_2(G)\,\delta^{ab}\int \frac{d^4p}{(2\pi)^4}\frac{(p+q)^\mu p^\nu\, \tilde{e}^2}{(p^2+i\epsilon)((p+q)^2+i\epsilon)}$$

This one is a little tougher. It's left as an exercise to simplify the loop momentum integral.

Fermion Vertex



$$\int \frac{d^4k}{(2\pi)^4}\,\tilde{e}^3\, T^b T^a T^b \frac{\gamma^\nu(\not{p}'+\not{k}+m)\gamma^\mu(\not{p}+\not{k}+m)\gamma_\nu}{\left((p'+k)^2-m^2+i\epsilon\right)\left((p+k)^2-m^2+i\epsilon\right)(p^2+i\epsilon)}$$

This is just as in QED except for the group theory factor which can be pulled out of the integral:

$$\begin{aligned} T^bT^aT^b &= T^bT^bT^a + T^b[T^a,T^b] \\ &= C_2(r_f)T^a + i\,T^b f^{abc}T^c \\ &= C_2(r_f)T^a + \tfrac{1}{2} i f^{abc}\, i f^{bcd}\, T^d \\ &= \left[C_2(r_f) - \tfrac{1}{2}C_2(G)\right]T^a \end{aligned}$$

r_f = rep. of gauge group under which fermion transforms.

<u>Note</u>: Except in the adjoint rep, $C_2(r_f) \neq \mu_{r_f}$

In the non-Abelian theory another diagram contributes to the Fermion Vertex Function!

This one is a little messy. It's left as an exercise.

Fermion Self Energy

There's just one 1-loop diagram (+ counterterm):

$$= \int \frac{d^4k}{(2\pi)^4} (-i\tilde{e})^2 \gamma^\mu T^a \frac{i((\not p+\not k)-m)}{(p+k)^2-m^2+i\epsilon} \gamma_\mu T^a \frac{(-i)}{k^2+i\epsilon}$$

$$= \tilde{e}^2 C_2(r_f)\cdot 2 \int \frac{d^4k}{(2\pi)^4} \frac{(\not p+\not k)+2m}{(p+k)^2-m^2+i\epsilon} \cdot \frac{1}{(k^2+i\epsilon)}$$