Non-Abelian Symmetries

We have studied the consequences of a number of continuous symmetries in this course. Some examples are the rotations in 3-dimensions, Lorentz transformations in 3+1 dimensions, the transformations $\psi \to e^{i\theta}\psi$ of a Dirac spinor.

Each of these classes of symmetry transformations share the mathematical properties of a group:

1. Closure: $\forall a,b \in G$, $ab \in G$
2. Associativity: $(ab)c = a(bc)$
3. $\exists$ Identity element: $\exists e \in G$ st $ea = ae = a$ $\forall a \in G$
4. $\exists$ Inverse element: $\forall a \in G$ $\exists a^{-1} \in G$ st $aa^{-1} = a^{-1}a = e$

Continuous groups can also be defined as smooth manifolds. Such groups are called Lie groups.

Commutative groups (eg. $\{e^{i\theta}\}$ w/ usual multiplication) are called Abelian.

Non-commutative groups (eg. rotations in $d>2$ dimensions) are called Non-Abelian.

In particle physics there are a number of approximate non-Abelian global symmetries (eg. isospin) and gauge invariances (color and weak interactions). To discuss the consequences of non-Abelian symmetries, and to build the Standard Model of Particle Physics, we first have to discuss non-Abelian groups.

Lie Groups and Lie Algebras

Examples:

Rotations	$SO(3)$	}
Spin	$SU(2)$	}
$\psi \to e^{i\theta}\psi$	$U(1)$	

The group describes how to transform elements of a vector space to other elements.

$$g : \Phi \to D(g)\,\Phi \equiv g\Phi, \quad g \in G = \text{a group of unitary } N{\times}N \text{ matrices.}$$

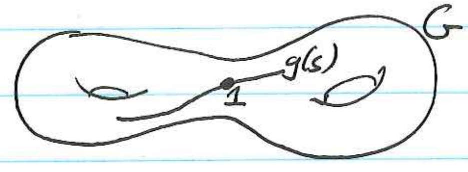
$\Phi \in N$-dim'l vector space.

Example: $\underbrace{\begin{pmatrix} x \\ y \end{pmatrix}}_{\Phi} \to \underbrace{\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}}_{g} \underbrace{\begin{pmatrix} x \\ y \end{pmatrix}}_{\Phi}, \quad g \in SO(2)$

$SO(2)$ = group of orthogonal $2{\times}2$ matrices w/ determinant 1.

Lie groups are parametrized by continuous parameters, like θ in $SO(2) \sim U(1)$.

Consider a path in group space that passes through the identity element $e \equiv 1$.



$g(s)$ = path in space of $N \times N$ unitary matrices st $g(0) = 1$.

Define $\boxed{\left.\frac{dg}{ds}\right|_{s=0} \equiv -iT} = N \times N$ matrix

g Unitary: $gg^\dagger = g^\dagger g = 1$

$$0 = \left.\frac{d}{ds}(gg^\dagger)\right|_{s=0} = g(0)\left.\frac{dg^\dagger}{ds}\right|_{s=0} + \left.\frac{dg}{ds}\right|_{s=0} g^\dagger(0)$$

$$= 1(iT^\dagger) + (-iT)\,1$$

$\rightarrow \boxed{T^\dagger = T}$ Hermitian

Parametrize elements of the group by x^a, $a = 1, \ldots, \dim G$.
s.t. $x^a = 0$ at $g(x^a) = 1$.

$$\left.\frac{dg}{ds}\right|_{s=0} = \sum_a \left.\frac{dx^a}{ds}\right|_{s=0} \frac{\partial g}{\partial x^a} \equiv -i \sum_a \frac{dx^a}{ds} T^a$$

The T^a's form a $(\dim G)$-dimensional vector space called $\mathcal{a}(G)$ = <u>Lie Algebra</u> of G.
The T^a's are called <u>generators</u> of G.

Consider $h\, g(s)\, h^{-1}$, $h \in G$.

$$h\, g(0)\, h^{-1} = h \cdot 1 \cdot h^{-1} = 1$$

$$i \frac{d}{ds}\left[h\, g(s)\, h^{-1}\right]\Big|_{s=0} = \boxed{h T h^{-1} \quad \in \mathcal{G}(G)}$$

Consider $i \frac{d}{ds}\left[h(s)\, T\, h^{-1}(s)\right]\Big|_{s=0}$, $h(s) \in G$ st $h(0) = 1$.

$$i \frac{d}{ds} h(s)\Big|_{s=0} = T' \quad \text{for some } T' \in \mathcal{G}(G)$$

So, $i \frac{d}{ds}\left[h(s)\, T\, h^{-1}(s)\right]\Big|_{s=0} = \boxed{[T', T] \in \mathcal{G}(G)}$

The Lie Algebra of G is specified by the commutation relations,

$$\boxed{[T^a, T^b] = i \underbrace{f^{abc}}_{\text{structure constants of } \mathcal{G}(G)} T^c} \quad , \quad a,b,c = 1, \ldots, \dim G$$

Choose a basis s.t. $\operatorname{Tr} T^a T^b = c\, \delta^{ab}$ for some c.

Then, $\frac{1}{c} \operatorname{Tr}\left([T^a, T^b] T^d\right) = \frac{i}{c} f^{abc} \operatorname{Tr}\left(T^c T^d\right) = \frac{i}{c} \cdot c\, f^{abc} \delta^{cd}$

$$\rightarrow \boxed{f^{abd} = -\frac{i}{c} \operatorname{Tr}\left([T^a, T^b] T^d\right)}$$

Structure constants are antisymmetric: $f^{abc} = -f^{bac}$, etc.

cyclic: $f^{abc} = f^{cab} = f^{bca}$

For any T^a, T^b, T^c there is an identity,

$$[T^a,[T^b,T^c]] + [T^b,[T^c,T^a]] + [T^c,[T^a,T^b]] = 0$$

$$\rightarrow \boxed{f^{ade} f^{bcd} + f^{bde} f^{cad} + f^{cde} f^{abd} = 0}$$ Jacobi Identity

Group elements from Algebra elements:
Infinitessimal deviation from identity

$$\Phi \rightarrow (1 + i\theta^a T^a)\Phi \quad , \theta^a \ll 1.$$

Φ complex: $\Phi^* \rightarrow (1 - i\theta^a (T^a)^*)\Phi^*$
$= (1 - i\theta^a (T^a)^T)\Phi^*$

Hence, the set of matrices $-(T^a)^T$ are the generators of a representation of G. If T^a are the generators of G in a representation r, then $-(T^a)^T$ are the generators of the conjugate representation to r.

Adjoint Representation $(T^b)_{ac} = i f^{abc}$

Jacobi ID $\rightarrow$ $[T^a, T^b]_{de} = i f^{abc} T^c_{de}$

f^{abc} — real, antisymmetric
$\rightarrow T^a = -(T^a)^T$ in adjoint rep $\rightarrow$ real representation.

Examples: $U(1)$ $\quad g = e^{i\theta}$, $\dim G = 1$.

$$\left.\frac{dg}{d\theta}\right|_{\theta=0} = i \rightarrow T = 1$$

$$[T,T] = 0 \rightarrow \text{Abelian group.}$$

$SU(2)$ = group of unitary 2×2 matrices w/ determinant 1.

$$g = e^{i\theta^a \sigma^a/2} \qquad \dim G = 3$$

σ^a = Pauli σ-matrices

$\rightarrow$ Fundamental representation of $SU(2)$

$\mathrm{Tr}\, T^a T^b = \frac{1}{2}\delta^{ab}$ in Fundamental rep.

↑ You can fix the constant arbitrarily, but once fixed for one rep, it's fixed (but not the same) for all reps.

$SU(3)$ = group of unitary 3×3 matrices w/ determinant 1.

$g = e^{i\theta^a T^a}$, $a = 1, \dots, 8$ $\quad \dim G = 8$

In fundamental rep, can choose basis:

Gell-Mann matrices

$$\left\{\begin{array}{l}
T^1 = \frac{1}{2}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad T^2 = \frac{1}{2}\begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad T^3 = \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \\
T^4 = \frac{1}{2}\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad T^5 = \frac{1}{2}\begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix}, \quad T^6 = \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \\
T^7 = \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \quad T^8 = \frac{1}{2\sqrt{3}}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}
\end{array}\right.$$