Functional Integral Quantization of Spinor Fields

Consider a Lagrangian of the form $\mathcal{L} = \Psi^\dagger A \Psi$, where $\Psi = (\psi_1, \ldots, \psi_n)$ are n fermionic fields, and $\Psi^\dagger = (\Psi^*)^T$. The matrix A includes terms independent of fields from the free Lagrangian, and terms dependent on bosonic fields.

For now, treat the bosonic fields as external background fields. We will integrate over the bosonic fields later.

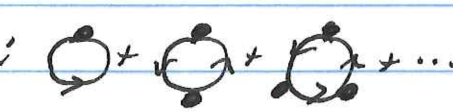
If Ψ were a collection of bosonic fields, we would write,

$$\langle 0, T=\infty | 0, T=-\infty \rangle \equiv e^{iW} = \underset{\text{normalization}}{N} \int d\Psi^* \, d\Psi \, e^{iS}$$

$$= N \int d\Psi^* \, d\Psi \, e^{i\int d^4x \, \Psi^\dagger A \Psi}$$

$$= \frac{1}{\det(iA)} = e^{\sum \text{connected vac-vac diagrams}}$$

Let  $\equiv$ all tree-level interactions

Vacuum-vacuum diagrams are all one loop: 

But recall that fermion loops all come with an extra minus sign, so $W \to -W$ if Ψ are fermions. Hence, we should get $\langle 0, T=\infty | 0, T=-\infty \rangle = \det(iA)$, not $\frac{1}{\det(iA)}$

This is in fact the result if the fermionic fields are <u>anticommuting complex functions</u> in the functional integral.

Anticommuting numbers $\equiv$ Grassmann numbers

<u>Integrals over Grassmann numbers</u>

$\int da\, f(a)$ should satisfy:

① Linearity: $\int da \left(f_1(a)\, c_1 + f_2(a)\, c_2 \right) = c_1 \int da\, f_1(a) + c_2 \int da\, f_2(a)$.

where c_1, c_2 = complex constants, and
$f_1(a), f_2(a)$ = functions over Grassmann a.

② Translation invariance: $\int da\, f(a+b) = \int da\, f(a)$
b = anticommuting constant.

Because $a^2 = 0$, any function of a has only a constant piece and a term linear in a. So we only need $\int da\, 1$ and $\int da\, a$.

$$\left.\begin{array}{ll} \text{Translation invariance} \rightarrow & \int da\,(a+b) = \int da\, a \\ \text{Linearity} \rightarrow & \int da\,(a+b) = \int da\, a + \left(\int da\right) b \end{array}\right\} \boxed{\int da\, 1 = 0}$$

Normalization: Define $\boxed{\int da\, a = 1.}$

The Grassmann integral table is therefore,

$$\int da \begin{Bmatrix} 1 \\ a \end{Bmatrix} = \begin{Bmatrix} 0 \\ 1 \end{Bmatrix}$$ — looks like differentiation!

<u>Grassmann integrals over two variables</u>, a and $\bar{a}$

$a^2 = 0$, $\bar{a}^2 = 0$, $\{a, \bar{a}\} = 0$.

Any function of $a, \bar{a}$ can be expanded in 4 monomials:
$1, a, \bar{a}, a\bar{a}$.

Convention: $\int da\, d\bar{a} \rightarrow$ Do $\bar{a}$ integral first, w/ a as constant.
Then do a integral.

Using the integral table for one Grassmann variable,

$$\int da\, d\bar{a} \begin{Bmatrix} 1 \\ a \\ \bar{a} \\ a\bar{a} \end{Bmatrix} = \int da\, d\bar{a} \begin{Bmatrix} 1 \\ a \\ \bar{a} \\ -\bar{a}a \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ -1 \end{Bmatrix}$$

"Gaussian": $\int d\bar{a}\, da\, e^{-\lambda \bar{a} a} \equiv \int d\bar{a}\, da\, (1 - \lambda \bar{a} a)$
$= -\lambda \underbrace{\int d\bar{a}\, da\, \bar{a} a}_{-1} = \lambda$

(Compare with $\int d\phi^* d\phi\, e^{-\alpha \phi^* \phi} \propto \frac{1}{\alpha}$)

Gaussian functional integral over Grassmann fields

$$\int d\bar{a}\, da\, e^{-\bar{a} A a}, \quad a = (a_1, \ldots, a_n)$$
$$\bar{a} = (\bar{a}_1, \ldots, \bar{a}_n)$$
$$d\bar{a}\, da = d\bar{a}_1 da_1 \cdots d\bar{a}_n da_n$$

Diagonalize A w/ linear transformation so that the integral takes the form

$$\left(\int d\bar{a}_1\, da_1\, e^{-\bar{a}_1 A_1 a_1}\right)\left(\int d\bar{a}_2\, da_2\, e^{-\bar{a}_2 A a_2}\right) \cdots,$$

where A_i = i^{th} eigenvalue of A.

Then, $\int d\bar{a}\, da\, e^{-\bar{a} A a} = \prod_{i=1}^{n} A_i = \det A.$

(Compare with $\int d\Phi^\dagger\, d\Phi\, e^{-\Phi^\dagger A \Phi} \propto \frac{1}{\det A}$)

Dirac Propagator

$$\langle 0| T[\Psi(x_1)\, \bar{\Psi}(x_2)]|0\rangle = \frac{\int D\bar{\Psi} D\Psi\, e^{i\int d^4x\, \bar{\Psi}(i\not{\partial}-m)\Psi}\, \Psi(x_1)\, \bar{\Psi}(x_2)}{\int D\bar{\Psi} D\Psi\, e^{i\int d^4x\, \bar{\Psi}(i\not{\partial}-m)\Psi}}$$

The denominator is $\int D\bar{\Psi}\, D\Psi\, \exp\left(i\int d^4x\, \bar{\Psi}(i\not{\partial}-m)\Psi\right)$

$$= \det(i\not{\partial}-m)$$

To calculate the numerator, use

$$\int \prod_i d\bar{a}_i \, da_i \; a_k \bar{a}_j \; \exp\left(-\sum_{\ell m} \bar{a}_\ell A_{\ell m} a_m\right)$$

$$= \int \prod_i d\bar{a}_i \, da_i \sum_{mn} \left(U_{km} a_m\right)\left(U^*_{jn} \bar{a}_n\right) \prod_\ell \left(1 - \bar{a}_\ell A_\ell a_\ell\right)$$

where A_ℓ = eigenvalues of A, and
$a = U^{-1} a$ diagonalizes A.

$$= \sum_{m,n} \left(\prod_{\ell \neq m} A_\ell\right) \delta_{mn} \; U_{km} U^*_{jn}$$

$$= \sum_{m,n} U_{km} \left(\frac{1}{A_m} \delta_{mn}\right) U^+_{nj} \quad (\det A)$$

$$= (A^{-1})_{kj} \; \det A \, .$$

Then, $\int \mathcal{D}\bar{\psi}\, \mathcal{D}\psi \; e^{i\int d^4x \, \bar{\psi}(i\not{\partial}-m)\psi} \; \psi(x_1)\, \bar{\psi}(x_2)$

$$= \det(i\not{\partial}-m) \left[-i(i\not{\partial}-m)\right]^{-1}_{(x_1, x_2)}$$

$$= \det(i\not{\partial}-m) \int \frac{d^4k}{(2\pi)^4} \frac{i\, e^{-ik\cdot(x_1-x_2)}}{\not{k} - m + i\epsilon}$$

$\uparrow$ from $\int_{-\infty - i\epsilon}^{\infty - i\epsilon} dt$, as in scalar case.

Hence, $\boxed{\langle 0 | T[\psi(x_1)\, \bar{\psi}(x_2)] | 0 \rangle = \int \frac{d^4k}{(2\pi)^4} \frac{i\, e^{-ik\cdot(x_1-x_2)}}{\not{k} - m + i\epsilon}}$,

which is the right answer.