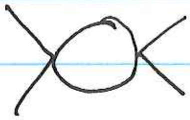
Minimal Subtraction

The physical renormalization prescription we have been using is not unique. There are alternatives which leave the result of loop calculations looking simpler, but the cost is that one has to work to obtain physical results.

Minimal Subtraction (MS) is a renormalization prescription that goes together with dimensional regularization. The scheme is to choose counterterms that cancel the poles as $d \to 4$, but that's it! You don't worry about physical masses and physical couplings; you just throw away the poles.

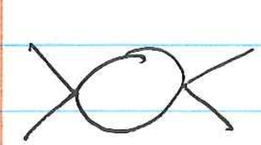
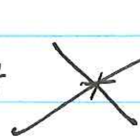
In the MS scheme we would have (Exercise)

 +  $\sim \frac{\lambda^2 \pi^2}{(2\pi)^4}\left[-\gamma_E + \log\left(\frac{\mu^2}{\pi q^2}\right)\right]$

(counterterm)

q = function of momenta, masses.

There is also a Modified Minimal Subtraction ($\overline{MS}$) scheme in which the unphysical finite constants are also dropped:

 +  $\sim \frac{\lambda^2 \pi^2}{(2\pi)^4} \log\left(\frac{\mu^2}{q^2}\right)$

Regulator Fields

We have introduced a hard momentum cutoff and dimensional regularization as techniques for regularizing divergences that appear at intermediate stages of Feynman amplitude calculations.

Another way to form convergent Feynman integrals is to modify the propagators in such a way that they equal the usual propagator at small K^2, but fall off quickly at large k^2.

One way to do this is to replace propagators by sums of other propagators suitably arranged to cancel terms in an expansion in k^2 about some large value which serves as the cutoff scale.

For example, the combination

$$\frac{i}{k^2-m^2+i\epsilon} - \frac{i}{k^2-M^2+i\epsilon} \sim \frac{1}{k^4} \quad \text{for } k^2 \gg M^2 \gg m^2.$$

If we need to make the propagator fall off even faster to make integrals convergent we can do that as follows:

$$\frac{i}{k^2-m^2+i\epsilon} \to \frac{i}{k^2-m^2+i\epsilon} + \sum_{n=1}^{N} \frac{i\, c_n^2}{k^2-M_n^2+i\epsilon}$$

not necessarily positive

At large k^2, $\frac{1}{k^2-m^2} \approx \frac{1}{k^2}\left(1+\frac{m^2}{k^2}+\left(\frac{m^2}{k^2}\right)^2+\cdots\right)$

If we choose $1+\sum_n c_n^2=0 \rightarrow \text{propagator} \sim \frac{1}{k^4}$

and $m^2+\sum_n c_n^2 M_n^2=0 \rightarrow \text{propagator} \sim \frac{1}{k^6}$

and $m^4+\sum_n c_n^2 M_n^4=0 \rightarrow \text{propagator} \sim \frac{1}{k^8}$

for $k^2 \gg \max(M_n^2)$

... etc.

This is like adding additional fields, but with the wrong sign of the propagator. One way to arrange that is to add the new fields w/ the correct sign kinetic and mass terms, but make the interactions a function of $\phi+\sum_n c_n \phi_n$. But some of the $c_n^2<0$, so this theory has non-Hermitian Hamiltonian.

Alternatively we can make the Hamiltonian Hermitian if the regulator fields have an inner product between states that is not positive definite, namely $\langle a|b\rangle_{new} = \langle a|(-1)^{N_{neg}}|b\rangle$, where N_{neg} is the number of particles of regulator fields w/ $c_n^2 < 0$.

Then the combination $\phi + \sum c_n \phi_n$ is Hermitian because for ϕ_n such that c_n is imaginary, $\phi_n^\dagger = -\phi_n$. The cost of Hermiticity is a non-positive definite, but conserved, probability distribution.

We can hope that this theory is still sensible for energies $\ll M_n$ because energy conservation will prevent the production of regulator particles in the final state.

A similar story goes through for fermions, i.e.

$$\frac{i}{\not{p}-m+i\epsilon} \rightarrow \frac{i}{\not{p}-m+i\epsilon} - \frac{i}{\not{p}-M+i\epsilon} \sim \frac{1}{p^2} \text{ at } p^2 \gg M^2.$$

Nice Properties of Regulator Fields:

- Preserve Lorentz Invariance
- Preserve global symmetries for massive particles
- Can accommodate gauge invariance

In practice, we can just use our integral table for integrands appearing in convergent combinations.

Pauli-Villars Regularization in QED

Pauli-Villars regularization is a variant of the regulator field approach to handling divergences. It is designed to maintain gauge invariance, and is an alternative to dimensional regularization.

Consider the QED Lagrangian, $\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \bar{\psi}(i\partial\!\!\!/ - eA\!\!\!/ - m)\psi$.
We introduce a set of vector fields $A_\mu^{(i)}$ w/ mass μ_i and Dirac fields $\psi^{(i)}$, mass M_i.
The new Lagrangian is,

$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{4}\sum_i F^{(i)}_{\mu\nu}F^{(i)\mu\nu} + \sum_i \mu_i^2 A^{(i)}_\mu A^{(i)\mu}$$

$$+ \bar{\psi}(i\partial\!\!\!/ - e(A\!\!\!/ + \sum_i c_i A\!\!\!/^{(i)}) - m)\psi$$

$$+ \sum_j \bar{\psi}^{(j)}(i\partial\!\!\!/ - e(A\!\!\!/ + \sum_i c_i A\!\!\!/^{(i)}) - M_j)\psi^{(j)}$$

The QED Lagrangian has a gauge invariance:

$$A_\mu \to A_\mu + \partial_\mu\theta(x)$$
$$\psi \to e^{-ie\theta(x)}\psi \quad , \quad \bar{\psi} \to e^{ie\theta(x)}\bar{\psi}$$

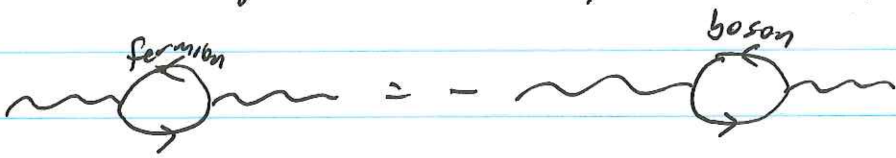
The QED + Regulator field Lagrangian is also gauge invariant:

$$A_\mu \to A_\mu + \partial_\mu\theta(x)$$
$$\psi \to e^{-ie\theta(x)}\psi, \quad \bar{\psi} \to e^{ie\theta(x)}\bar{\psi}$$
$$A_\mu^{(i)} \to A_\mu^{(i)}$$
$$\psi^{(i)} \to e^{-ie\theta(x)}\psi^{(i)}, \quad \bar{\psi}^{(i)} \to e^{ie\theta(x)}\bar{\psi}^{(i)}$$

It is not immediately obvious how these regulator fields will cure divergences, for example in the photon self energy diagram  .

The highest divergence is quadratic, and is independent of the Dirac field masses. Unlike in our earlier discussion of regulator fields, to preserve gauge invariance we could not simply replace $\psi \to \psi + \sum_i c_i \psi^{(i)}$ in the Lagrangian.

The answer is that we can make some of the $\psi^{(i)}$'s bosons, despite transforming as Dirac spinors. Bosons don't get a minus sign for the spinor trace, so



for spinor fields of same mass but opposite statistics.

To regularize the divergences in the photon self energy we need at least three Dirac spinor regulator fields. If $\psi^{(1)}$, $\psi^{(2)}$ are bosons and $\psi^{(3)}$ is a fermion, then the highest mass-independent divergence is eliminated.

The next highest divergence is $\propto m^2 - M_1^2 - M_2^2 + M_3^2$, which can be chosen to vanish with arbitrarily large M_1, M_2, M_3.

The photon field is replaced by $A^\mu + \sum_i c_i A_i^\mu$ in interactions, as in the ordinary regulator field approach. Hence, diagrams including photon propagators can be made more convergent by effectively modifying the propagators at large k^2.

To summarize, for photons divergences cancel on internal lines; for charged particles divergences cancel on loops.