Electroweak Interactions - The Standard Model

The electroweak interactions are governed by an $SU(2) \times U(1)$ gauge theory. The Higgs mechanism spontaneously breaks the gauge group to a $U(1)$ subgroup. The corresponding massless gauge field is identified w/ the photon, and mediates the electromagnetic interaction. The three massive gauge bosons are called the $W^{\pm}$ and the Z.

Note: The $U(1)$ factor in $SU(2) \times U(1)$ describes hypercharge, and is not the same as the unbroken $U(1)$ describing electromagnetism. We will write $SU(2) \times U(1)_Y$ and $U(1)_{em}$ to distinguish the $U(1)$'s.

In the Standard Model there is a complex doublet of scalar fields with hypercharge $Y(\Phi) = 1$.

$$\Phi = \begin{pmatrix} \phi^+ \\ \phi^0 \end{pmatrix} \quad , \quad \phi^+, \phi^0 \text{ complex scalar fields.}$$

Call the $SU(2)$ gauge bosons A_μ^a, gauge coupling g
$U(1)_Y$ gauge boson B_μ, gauge coupling $\frac{g'}{2}$.

$$D_\mu \Phi = \left(\partial_\mu - i \frac{g}{2} \sigma^a A_\mu^a - i \frac{g'}{2} B_\mu \right) \Phi$$

$$\mathcal{L} \supset (D_\mu \Phi)^\dagger (D^\mu \Phi) - V(\Phi)$$

$$V(\Phi) = -\mu^2 \Phi^\dagger \Phi + \lambda (\Phi^\dagger \Phi)^2 , \quad \lambda, \mu > 0.$$

Choose $\langle \Phi \rangle_0 \equiv \langle 0 | \Phi | 0 \rangle = \begin{pmatrix} 0 \\ v/\sqrt{2} \end{pmatrix}$, $v = \left(\frac{\mu^2}{\lambda} \right)^{1/2}$

Write $\Phi(x) = U^{-1}(\xi^a(x)) \begin{pmatrix} 0 \\ \frac{v + \eta(x)}{\sqrt{2}} \end{pmatrix}$

where $U(\xi^a) = \exp\left(i \xi^a \frac{\sigma^a}{2v} \right)$.

Transform to Unitary gauge:

$$\begin{cases} \Phi'(x) = U(\xi^a(x))\, \Phi(x) = \begin{pmatrix} 0 \\ \frac{v + \eta(x)}{\sqrt{2}} \end{pmatrix} , \quad \boxed{\eta(x) = \text{Higgs field}} \\ A'^a_\mu \frac{\sigma^a}{2} = U(\xi^a(x))\, A^a_\mu \frac{\sigma^a}{2} U^{-1}(\xi^a(x)) + \frac{i}{g} U \partial_\mu U^{-1} \end{cases}$$

$$D_\mu \Phi' = U D_\mu \Phi = U \left(\partial_\mu - \frac{ig}{2} \sigma^a A^a_\mu - \frac{ig'}{2} B_\mu \right) \underset{U^{-1}U}{\Phi}$$

$$= \left(\partial_\mu - \frac{ig}{2} \sigma^a A'^a_\mu - \frac{ig'}{2} B_\mu \right) \Phi'$$

The scalar field potential is:

$$V(\Phi') = -\frac{\mu^2}{2} (v + \eta(x))^2 + \frac{\lambda}{4} (v + \eta(x))^4$$

$$= \text{const.} + \left(-\frac{\mu^2}{2} \eta^2 + \frac{3}{2} \lambda v^2 \eta^2 \right) + \lambda v \eta^3 + \frac{\lambda}{4} \eta^4$$

$$= \text{const.} + \mu^2 \eta^2 + \lambda v \eta^3 + \frac{\lambda}{4} \eta^4$$

$\boxed{m_\eta = \sqrt{2}\, \mu}$ Higgs mass

Gauge Field Mass Terms:

$$\mathcal{L} \supset \frac{1}{2}(0, v)\left(\frac{g}{2}\sigma^a A'^a_\mu + \frac{g'}{2}B_\mu\right)\left(\frac{g}{2}\sigma^b A'^{\mu b} + \frac{g'}{2}B^\mu\right)\begin{pmatrix}0\\v\end{pmatrix}$$

$$= \frac{v^2}{8}g^2\left((A'^1_\mu)^2 + (A'^2_\mu)^2\right) + \frac{v^2}{8}\left(gA'^3_\mu - g'B_\mu\right)^2$$

$$\equiv M_W^2 W^+_\mu W^{-\mu} + \frac{1}{2}M_Z^2 Z_\mu Z^\mu$$

where $W^\pm_\mu = (A'^1_\mu \mp i A'^2_\mu)/\sqrt{2}$, $\boxed{M_W^2 = \frac{g^2 v^2}{4}}$

$$\frac{1}{2}M_Z^2 Z_\mu Z^\mu = \frac{v^2}{8}\left(gA'^3_\mu - g'B_\mu\right)^2$$

$$= \frac{v^2}{8}(g^2+g'^2)\left(\frac{gA'^3_\mu - g'B_\mu}{\sqrt{g^2+g'^2}}\right)^2$$

For canonically normalized kinetic term.

$$\Rightarrow \boxed{M_Z^2 = \frac{v^2}{4}(g^2+g'^2)}$$

☆ Define $\cos\theta_W = \frac{g}{\sqrt{g^2+g'^2}}$, $\sin\theta_W = \frac{g'}{\sqrt{g^2+g'^2}}$

weak mixing angle, sometimes called the Weinberg angle.

$$Z_\mu = \cos\theta_W A'^3_\mu - \sin\theta_W B_\mu$$

Orthogonal combination: $A_\mu = \sin\theta_W A'^3_\mu + \cos\theta_W B_\mu$

The normalization is chosen so that

$$(\partial_\mu A'^3_\nu - \partial_\nu A'^3_\mu)^2 + (\partial_\mu B_\nu - \partial_\nu B_\mu)^2$$

$$= (\partial_\mu Z_\nu - \partial_\nu Z_\mu)^2 + (\partial_\mu A_\nu - \partial_\nu A_\mu)^2$$

☆ A_μ has no mass term $\longrightarrow$ <u>Photon</u>

The weak mixing angle will appear in couplings to fermions → measured in scattering experiments

Tree-level Standard Model prediction:

$$\boxed{\rho \equiv \frac{M_W^2}{M_Z^2 \cos^2\theta_W} = 1}$$

The ρ parameter.

Custodial Symmetry in the Higgs Sector

The Higgs potential is of the form $V(\Phi) = V(\Phi^\dagger \Phi)$.
In terms of the real and imaginary parts of Φ,

$$\Phi = \begin{pmatrix} \phi^+ \\ \phi^0 \end{pmatrix} = \begin{pmatrix} (\phi_1^+ + i\phi_2^+)/\sqrt{2} \\ (\phi_1^0 + i\phi_2^0)/\sqrt{2} \end{pmatrix}$$

$$\Phi^\dagger \Phi = \frac{1}{2}\left((\phi_1^+)^2 + (\phi_2^+)^2 + (\phi_1^0)^2 + (\phi_2^0)^2 \right)$$

Hence, the potential has an $O(4) \sim SU(2) \times SU(2)$ global symmetry.
The Higgs VEV breaks $O(4) \to O(3) \sim SU(2) \equiv$ custodial symmetry
The $SU(2)$ gauge fields $A_\mu'^a$, $a=1,2,3$ are a degenerate triplet under the unbroken $O(3)$, i.e. the mass terms have the form

$$\mathcal{L} \supset \frac{1}{2} M_W^2 \left[(A_\mu'^1)^2 + (A_\mu'^2)^2 + (A_\mu'^3)^2 \right]$$

The trace of the (A_μ^3, B) mass matrix $= \frac{M_Z^2}{2} + \frac{M_\gamma^2}{2} = \frac{M_Z^2}{2}$
The determinant is $\frac{M_Z^2}{2} \cdot \frac{M_\gamma^2}{2} = 0$. This determines the mass matrix:

$$\mathcal{L} \supset \frac{1}{2} (A_\mu'^3, B_\mu) \begin{pmatrix} M_W^2 & M_W (M_Z^2 - M_W^2)^{1/2} \\ M_W (M_Z^2 - M_W^2)^{1/2} & M_Z^2 - M_W^2 \end{pmatrix} \begin{pmatrix} A'^{\mu 3} \\ B^\mu \end{pmatrix}$$

Diagonalizing the mass matrix, it immediately follows that

$$\cos\theta_W = \frac{M_W^2}{\left(M_W^4 + M_W^2(M_Z^2 - M_W^2)\right)^{1/2}} = \frac{M_W}{M_Z}$$

Hence, the prediction $\rho = 1$ is a consequence of the custodial symmetry in the Higgs sector.

Fermion Gauge Couplings

Consider a single generation of fermions:

$Q_L = \begin{pmatrix} u_L \\ d_L \end{pmatrix}$, $Y(Q_L) = 1/3$ ← Define $Q_L' = U(\xi^a) Q_L$

$L_L = \begin{pmatrix} \nu_L \\ e_L \end{pmatrix}$ $Y(L_L) = -1$ ← Define $L_L' = U(\xi^a) L_L$

u_R $Y(u_R) = 4/3$

d_R $Y(d_R) = -2/3$

e_R $Y(e_R) = -2$

(ν_R $Y(\nu_R) = 0$)

↑ Hyperchanges

$$\mathcal{L} \supset \bar{L}_L' \left(\frac{g}{2}\sigma^a \not{A}'^a - \frac{g'}{2}\not{B}\right) L_L' + \bar{Q}_L' \left(\frac{g}{2}\sigma^a \not{A}'^a + \frac{g'}{6}\not{B}\right) Q_L'$$

$$- \bar{e}_R\, g' \not{B}\, e_R + \bar{u}_R \frac{2g'}{3}\not{B}\, u_R - \bar{d}_R \frac{g'}{3}\not{B}\, d_R$$

Charged Current: $\mathcal{L}_{CC} = \bar{L}_L' \left(\frac{g}{2}\sigma^1 \not{A}'^1 + \frac{g}{2}\sigma^2 \not{A}'^2\right) L_L'$
$+ \bar{Q}_L' \left(\frac{g}{2}\sigma^1 \not{A}'^1 + \frac{g}{2}\sigma^2 \not{A}'^2\right) Q_L'$

$$\mathcal{L}_{CC} = \frac{g}{2}(\bar{\nu}_L', \bar{e}_L')\begin{pmatrix} 0 & A'^1 - iA'^2 \\ A'^1 + iA'^2 & 0 \end{pmatrix}\begin{pmatrix} \nu_L' \\ e_L' \end{pmatrix}$$

$$+ \frac{g}{2}(\bar{u}_L', \bar{d}_L')\begin{pmatrix} 0 & A'^1 - iA'^2 \\ A'^1 + iA'^2 & 0 \end{pmatrix}\begin{pmatrix} u_L' \\ d_L' \end{pmatrix}$$

$$= \frac{g}{\sqrt{2}}\left(\bar{\nu}_L'\gamma^\mu e_L' + \bar{u}_L'\gamma^\mu d_L'\right)W_\mu^+$$

$$+ \frac{g}{\sqrt{2}}\left(\bar{e}_L'\gamma^\mu \nu_L' + \bar{d}_L'\gamma^\mu u_L'\right)W_\mu^-$$

$$\equiv \frac{g}{\sqrt{2}}\left(J^{+\mu}W_\mu^+ + J^{-\mu}W_\mu^-\right) \quad \text{, where}$$

$$\boxed{\begin{aligned} J^{+\mu} &= \bar{\nu}_L'\gamma^\mu e_L' + \bar{u}_L'\gamma^\mu d_L' \\ J^{-\mu} &= \bar{e}_L'\gamma^\mu \nu_L' + \bar{d}_L'\gamma^\mu u_L' \end{aligned}}$$

Neutral Currents

$$\mathcal{L}_{NC} = \bar{L}_L'\left(\frac{g}{2}\sigma^3 \not{A}'^3 - \frac{g'}{2}\not{B}\right)L_L' + \bar{Q}_L'\left(\frac{g}{2}\sigma^3\not{A}'^3 + \frac{g'}{6}\not{B}\right)Q_L'$$
$$- \bar{e}_R g'\not{B} e_R + \bar{u}_R \frac{2g'}{3}\not{B} u_R - \bar{d}_R \frac{g'}{3}\not{B} d_R$$

$$= \bar{\nu}_L'\left(\frac{g}{2}\not{A}'^3 - \frac{g'}{2}\not{B}\right)\nu_L' - \bar{e}_L'\left(\frac{g}{2}\not{A}'^3 + \frac{g'}{2}\not{B}\right)e_L'$$

$$+ \bar{u}_L'\left(\frac{g}{2}\not{A}'^3 + \frac{g'}{6}\not{B}\right)u_L' + \bar{d}_L'\left(-\frac{g}{2}\not{A}'^3 + \frac{g'}{6}\not{B}\right)d_L'$$

$$- \bar{e}_R g'\not{B} e_R + \bar{u}_R \frac{2g'}{3}\not{B} u_R - \bar{d}_R \frac{g'}{3}\not{B} d_R$$

$\mathcal{L}_{NC}$ can be written $\mathcal{L}_{NC} = \sum_{\substack{\text{chiral} \\ \text{fermions} \\ f}} \bar{f}\left(g T^3 \not{A}'^3 + \frac{g'}{2} Y(f) \not{B}\right) f$

where $T^3 = \frac{\sigma^3}{2}$ for the left-handed doublets
$T^3 = 0$ for the right-handed singlets
$Y(f)$ = hypercharge of fermion f.

Recall
$$\begin{pmatrix} Z_\mu \\ A_\mu \end{pmatrix} = \begin{pmatrix} \cos\theta_W & -\sin\theta_W \\ \sin\theta_W & \cos\theta_W \end{pmatrix} \begin{pmatrix} A'^3_\mu \\ B_\mu \end{pmatrix}$$

Hence,
$$\begin{pmatrix} A'^3_\mu \\ B_\mu \end{pmatrix} = \begin{pmatrix} \cos\theta_W & \sin\theta_W \\ -\sin\theta_W & \cos\theta_W \end{pmatrix} \begin{pmatrix} Z_\mu \\ A_\mu \end{pmatrix}$$

$$\mathcal{L}_{NC} = \sum_f \bar{f}\left(g T^3 \left(\cos\theta_W \not{Z} + \sin\theta_W \not{A}\right) + \frac{g'}{2} Y(f)\left(-\sin\theta_W \not{Z} + \cos\theta_W \not{A}\right)\right) f$$

$$= \sum_f \Big[\bar{f}\left(g T^3 \cos\theta_W - \frac{g'}{2} Y(f) \sin\theta_W\right) \not{Z} f$$

$$+ \bar{f}\left(g T^3 \sin\theta_W + \frac{g'}{2} Y(f) \cos\theta_W\right) \not{A} f \Big]$$

Recall $\cos\theta_W = \frac{g}{(g^2+g'^2)^{1/2}}$, $\sin\theta_W = \frac{g'}{(g^2+g'^2)^{1/2}}$

Define $\boxed{e = g \sin\theta_W = g' \cos\theta_W}$

The electromagnetic interaction takes the form

$$\boxed{\mathcal{L}_{em} = \sum_f e\, \bar{f}\left(T^3 + \tfrac{1}{2} Y(f)\right) \gamma^\mu f\, A_\mu}$$

$$\equiv e J^\mu_{em} A_\mu, \quad \boxed{J^\mu_{em} = \sum_f \bar{f}\left(T^3 + \tfrac{1}{2} Y(f)\right) \gamma^\mu f}$$

We therefore identify the electric charge

$$\boxed{Q = T^3 + \tfrac{1}{2} Y}$$

Fermion	T^3	Y	$Q = T^3 + Y/2$
u_L	$\frac{1}{2}$	$\frac{1}{3}$	$2/3$
d_L	$-\frac{1}{2}$	$\frac{1}{3}$	$-1/3$
ν_L	$\frac{1}{2}$	-1	0
e_L	$-\frac{1}{2}$	-1	-1
u_R	0	$4/3$	$2/3$
d_R	0	$-2/3$	$-1/3$
e_R	0	-2	-1
(ν_R	0	0	0)

Z couplings

$$\mathcal{L}_Z = \sum_f \bar{f}\left(g T^3 \cos\theta_W - \frac{g'}{2} Y(f) \sin\theta_W\right)\gamma^\mu f \, Z_\mu$$

$$= \sum_f \bar{f}\left[\frac{g}{\cos\theta_W}(1-\sin^2\theta_W)T^3 - \frac{g\sin^2\theta_W}{\cos\theta_W}\frac{Y(f)}{2}\right]\gamma^\mu f \, Z_\mu$$

using $g'\cos\theta_W = g\sin\theta_W$

$$= \sum_f \bar{f}\frac{g}{\cos\theta_W}\left[T^3 - \sin^2\theta_W \, Q(f)\right]\gamma^\mu f \, Z_\mu$$

$$\equiv \frac{g}{\cos\theta_W} J_0^\mu Z_\mu, \text{ where } J_0^\mu \equiv \sum_f \left[g_L^f \bar{f}_L \gamma^\mu f_L + g_R^f \bar{f}_R \gamma^\mu f_R\right]$$

$$\boxed{g_{L,R}^f = T^3(f_{L,R}) - \sin^2\theta_W \, Q(f)}$$

weak neutral-current couplings.

Fermion	T^3	Q	g^f
u_L	$1/2$	$2/3$	$\frac{1}{2} - \frac{2}{3}\sin^2\theta_W$
d_L	$-1/2$	$-1/3$	$-\frac{1}{2} + \frac{1}{3}\sin^2\theta_W$
ν_L	$1/2$	0	$\frac{1}{2}$
e_L	$-1/2$	-1	$-\frac{1}{2} + \sin^2\theta_W$
u_R	0	$2/3$	$-\frac{2}{3}\sin^2\theta_W$
d_R	0	$-1/3$	$\frac{1}{3}\sin^2\theta_W$
e_R	0	-1	$\sin^2\theta_W$
$(\nu_R$	0	0	0