Physics 722, Spring 2019
Problem Set 8
Due Tuesday, April 4.

1. *Lie Groups*

a) Show that if the generators of a Lie group satisfy the Lie algebra,

$$\left[T^a, T^b\right] = i\, f^{abc}\, T^c,$$

then the generators in the adjoint representation, defined by

$$(T^a)_{bc} = -i\, f^{abc},$$

satisfy the Lie algebra.

b) Show that $T^2 = T^a T^a$ commutes with all the group generators. As a consequence, $(T^2)_{ij} = C_2(r)\, \delta_{ij}$, where the constant $C_2(r)$ is called the *quadratic Casimir* of the representation r.

c) Suppose the generators are normalized so that

$$\mathrm{Tr}\, T^a T^b = C(r)\, \delta^{ab},$$

where the constant $C(r)$ depends on the representation r. Suppose the generators in the representation r are $d(r) \times d(r)$ matrices. Show that

$$d(r) C_2(r) = d(G) C(r),$$

where G stands for the adjoint representation.

d) Suppose the generators of SU(N) in the fundamental representation are normalized by

$$\mathrm{Tr}\, T^a T^b = \frac{1}{2} \delta^{ab}.$$

Calculate the quadratic Casimir C_2 in this representation.

e) Suppose the generators of SU(2) in the fundamental representation are normalized as in part (d). What is the quadratic Casimir C_2 in the d-dimensional representation of SU(2) as a function of d?

Challenge: Assume the generators of SU(N) in the fundamental representation are as in part (d). Show that in the adjoint representation $C(G) = C_2(G) = N$.

2. *Coulomb gauge*

Coulomb gauge is defined by the condition $\nabla \cdot \mathbf{A} = 0$. The Fadeev-Popov determinant is a constant in this case, as in the covariant gauges discussed in class. The determinant can then be absorbed in the normalization of the functional integral.

a) Show that the spatial components of the photon propagator are,

$$\frac{-i}{k^2 + i\epsilon}\left(g_{ij} + \frac{k_i k_j}{\mathbf{k}^2}\right).$$

b) Show that the 0-0 component of the photon propagator is,

$$\frac{i}{\mathbf{k}^2}.$$

c) What are the time-space components of the photon propagator?