Physics 722, Spring 2019
Problem Set 7
Due Thursday, March 28.

1. *Functional Integral for Complex Scalars*

Using the functional integral for a free complex scalar field ϕ with mass m, evaluate the following correlation functions:

$$\langle 0|\phi(x)|0\rangle, \ \langle 0|T\left[\phi(x)\phi(y)\right]|0\rangle, \ \langle 0|T\left[\phi(x)\overline{\phi}(y)\right]|0\rangle.$$

2. *Potential Energy from the Functional Integral*

Consider a real scalar field $\phi(x)$ coupled to a static background source $\rho(\mathbf{x})$, with action

$$S[\rho(\mathbf{x})] = \int d^4x \left[\frac{1}{2}\left(\partial_\mu\phi\right)^2 - \frac{m^2}{2}\phi^2 - \rho(\mathbf{x})\phi(x)\right].$$

It follows from our discussion of the functional integral that

$$\begin{aligned} \lim_{T\to\infty(1-i\epsilon)} e^{-i\Delta E_0 T} &\equiv \lim_{T\to\infty(1-i\epsilon)} \frac{\langle 0|\exp[-iH[\rho(\mathbf{x})]T]|0\rangle}{\langle 0|\exp[-iH[0]T]|0\rangle} \\ &= \frac{\int \mathcal{D}\phi \, \exp\left[iS[\rho(\mathbf{x})]\right]}{\int \mathcal{D}\phi \, \exp\left[iS[0]\right]}. \end{aligned}$$

From this relation, calculate the the ground state energy in the theory with source $\rho(\mathbf{x})$, relative to that of the source-free theory, which should take the form

$$\Delta E_0 = \frac{1}{2}\int d^3x \, d^3x' \, \rho(\mathbf{x}')V(\mathbf{x}' - \mathbf{x})\rho(\mathbf{x}).$$

You should evaluate $V(\mathbf{x}' - \mathbf{x})$. Don't just leave it in the form of a Fourier transform.

Hint: Think about Gaussian integrals with a linear term in addition to the quadratic term in the argument of the exponential. Find a transformation from $\phi(\mathbf{x})$ to some $\tilde{\phi}(\mathbf{x})$ that would "complete the square" in the functional integral.