Physics 722, Spring 2019
Problem Set 4
Due Thursday, February 28.

1. *Gamma Matrices in Even Dimensions*
In dimensional regularization we analytically continue in the number of spacetime dimensions. It is natural to consider what the gamma matrices look like in other than four (integer) dimensions.

In d spacetime dimensions, we want to find d matrices satisfying:

$$\gamma_0^\dagger = \gamma_0, \quad \gamma_i^\dagger = -\gamma_i, \quad i = 1, \ldots, d-1,$$
$$\{\gamma_\mu, \gamma_\nu\} = 2g_{\mu\nu}, \quad \mu, \nu = 0, \ldots, d-1.$$

It's easier to construct matrices satisfying,

$$\gamma_\mu^\dagger = \gamma_\mu, \quad \mu = 1, \ldots, d,$$
$$\{\gamma_\mu, \gamma_\nu\} = 2\delta_{\mu\nu}, \quad \mu, \nu = 1, \ldots, d.$$

You can get the gamma matrices you want from these by letting $\gamma_d \to \gamma_0, \ \gamma_j \to i\gamma_j$.

a) Assume d is even. Define,

$$\begin{aligned} a_1 &= \frac{1}{2}(\gamma_1 + i\gamma_2), \\ a_2 &= \frac{1}{2}(\gamma_3 + i\gamma_4), \\ &\vdots \\ a_{d/2} &= \frac{1}{2}(\gamma_{d-1} + i\gamma_d), \end{aligned}$$

where the gamma matrices in these expressions satisfy the second set of conditions above.

Show that,

$$\{a_i, a_j\} = \{a_i^\dagger, a_j^\dagger\} = 0,$$
$$\{a_i, a_j^\dagger\} = \delta_{ij} \quad i, j = 1, \ldots, d/2.$$

This is the algebra of raising and lowering operators for $d/2$ independent two-level systems.

b) In two dimensions, construct a matrix representation for a and $a^\dagger$. What are γ_1 and γ_2 in that representation?

c) In d even dimensions, what is the dimensionality of your representation of the gamma matrices? Evaluate $\mathrm{Tr}\, 1$ and $\mathrm{Tr}\, \not{p}\not{p}$ in that representation.

d) Check that $\prod_i \gamma_i$ anticommutes with all of the γ_i. This is the analog of γ_5 in any even dimension.

Comment: In odd dimensions the first $d-1$ gamma matrices can be constructed as above, and $\gamma_d = \pm\gamma_1\gamma_2 \cdots \gamma_{d-1}$ completes the gamma matrix algebra. There are two independent representations of the gamma matrix algebra in odd dimensions, differing in the sign of γ_d. These representations are exchanged by parity, and both representations appear in a parity-conserving theory.

2. *Rosenbluth Formula*
This is problem 6.1 in Peskin and Schroeder.

Assume the vertex function for the proton is of the form,

$$\tilde{\Gamma}^\mu(p', p) = \gamma^\mu\, F_1(q^2) + \frac{i\,\sigma^{\mu\nu}\, q_\nu}{2m}\, F_2(q^2),$$

where p, p' are the ingoing, outgoing proton momenta, and $q = p' - p$ is the ingoing photon momentum, and $\sigma^{\mu\nu} = i/2\,[\gamma^\mu, \gamma^\nu]$. (A factor of electric charge e is factored out of $\tilde{\Gamma}$ as for the electron.)

The form factors for strongly interacting particles like the proton are generally difficult to calculate, but they can be determined experimentally. Consider scattering of an energetic electron with energy $E \gg m_e$ from a proton initially at rest. To leading order in e the electron vertex function can be approximated by the tree-level interaction vertex, while the proton electromagnetic form factors $F_1(q^2)$ and $F_2(q^2)$ contain information about the strong interactions.

At leading order in $\alpha = e^2/4\pi$, show that the elastic scattering cross section takes the Rosenbluth form,

$$\frac{d\sigma}{d\cos\theta} = \frac{\pi\alpha^2\left[\left(F_1^2 - \frac{q^2}{4m^2}F_2^2\right)\cos^2\frac{\theta}{2} - \frac{q^2}{2m^2}\left(F_1 + F_2\right)^2 \sin^2\frac{\theta}{2}\right]}{2E^2\left[1 + \frac{2E}{m}\sin^2\frac{\theta}{2}\right]\sin^4\frac{\theta}{2}},$$

where θ is the lab frame scattering angle and F_1 and F_2 are evaluated at the momentum transfer q^2 associated with elastic scattering at this angle.