Physics 722, Spring 2019
Problem Set 3, due Thursday, February 14.

1. *Derivative Interactions*

In class we mentioned the subtleties in dealing with derivative interactions. Here you will study a simple example in which the naive handling of derivatives gives the correct answer.

Consider a free real scalar field with Lagrangian,

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi)^2 - \frac{m^2}{2}\phi^2.$$

We can equivalently write this as a Lagrangian in terms of a rescaled field $\tilde{\phi} \equiv Z^{-1/2}\phi$ as,

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \tilde{\phi})^2 - \frac{m^2}{2}\tilde{\phi}^2 + (Z-1)\left[\frac{1}{2}(\partial_\mu \tilde{\phi})^2 - \frac{m^2}{2}\tilde{\phi}^2\right].$$

The Fourier-transformed two-point function for the field ϕ is the usual free-field propagator:

$$\int d^4x\, e^{ik\cdot x}\langle 0|T\left(\phi(x)\phi(0)\right)|0\rangle = \frac{i}{k^2 - m^2 + i\epsilon}.$$

a) Given the relationship between $\tilde{\phi}$ and ϕ, the propagator for the field $\tilde{\phi}$ is a simple rescaling of the propagator for the field ϕ. What is

$$\int d^4x\, e^{ik\cdot x}\langle 0|T\left(\tilde{\phi}(x)\tilde{\phi}(0)\right)|0\rangle?$$

In the remainder of this problem you will reproduce the result of part (a) by evaluating $\langle 0|T\left(\tilde{\phi}(x)\tilde{\phi}(0)\right)|0\rangle$ in perturbation theory, where you are to think of the the terms in the Lagrangian proportional to $(Z-1)$ as being interaction terms.

b) Derive the Feynman rules for the two-point vertices corresponding to the interactions in this theory, treating the derivatives in the interactions naively.

c) By summing over *all* diagrams that contribute to the Fourier transformed two-point function, $\int d^4x\, e^{ik\cdot x}\langle 0|T\left(\tilde{\phi}(x)\tilde{\phi}(0)\right)|0\rangle$, show that you recover the result of part (a).

2. *Fermion self energy*

Consider the theory of a fermion $\psi(x)$ with mass m, Yukawa coupled to a real scalar field $\phi(x)$ with mass μ:

$$\mathcal{L} = \overline{\psi}(i\partial\!\!\!/ - M)\psi + \frac{1}{2}(\partial_\mu\phi)^2 - \frac{m^2}{2}\phi^2 - g\overline{\psi}\psi\phi - \frac{\lambda_3}{3!}\phi^3 - \frac{\lambda_4}{4!}\phi^4 + \mathcal{L}_{\mathrm{CT}}.$$

a) Calculate the one-loop renormalized fermion self energy $\widetilde{\Sigma}(p\!\!\!/)$. The renormalized self energy should satisfy $\widetilde{\Sigma}(M) = 0$ and $d\widetilde{\Sigma}/dp\!\!\!/\,|_{p\!\!\!/=M} = 0$. Use a hard momentum cutoff to regularize any divergent integrals appearing at intermediate stages of the calculation, and check that those divergences are cancelled in the renormalization procedure. Your result should be left in terms of integral(s) over a single Feynman parameter.

b)Does $\widetilde{\Sigma}(p\!\!\!/)$ have a branch cut? If so, what is the physical interpretation of the value of p^2 at the branch point (not at infinity)?

3. *Volume of a d-dimensional sphere*

a) Compute the $(d+1)$-dimensional integral

$$I_{d+1} \equiv \int d^{d+1}x\, e^{-(x_1^2+x_2^2+\cdots+x_{d+1}^2)}.$$

b) Write the integral in spherical coordinates and separate out the angular part of the integral. By a suitable change of variables, use the definition of the gamma function,

$$\int_0 x^{t-1}e^{-x}\,dx,$$

to evaluate the integral in terms of the gamma function and the volume of a d-dimensional unit sphere.

c) By comparing the results of parts (a) and (b), determine the volume of the d-dimensional unit sphere.