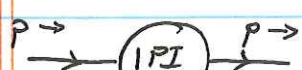
Fermion Self Energy

We define the fermion self energy as a sum over 1PI diagrams as for scalar fields:



$$\equiv \frac{i(\not{p}+m)}{p^2-m^2+i\epsilon}\left(-i\,\tilde{\Sigma}(p)\right)\frac{i(\not{p}+m)}{p^2-m^2+i\epsilon}$$

$\tilde{\Sigma}(p)$ = renormalized fermion self energy, contains contributions from counterterms; satisfies renormalization conditions to be determined.

Counterterms contributing to $\tilde{\Sigma}(p)$ at lowest order in the couplings: $\mathcal{L}_{CT} \supset D\bar{\tilde{\Psi}}i\not{\partial}\tilde{\Psi} - E\bar{\tilde{\Psi}}\tilde{\Psi}$

$$iD\not{p} - iE$$

$\tilde{\Sigma}(p)$ is a 4×4 matrix function of the momentum p^μ. By Lorentz invariance,

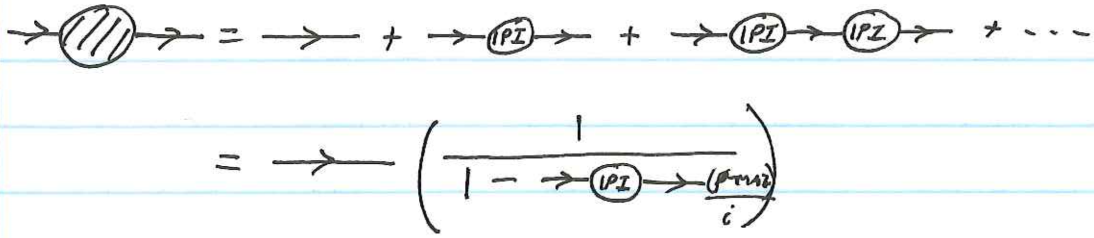
$$\tilde{\Sigma}(p) = a(p^2) + b(p^2)\gamma_5 + c(p^2)\not{p} + d(p^2)\gamma_5\not{p} + \underbrace{e(p^2)\sigma_{\mu\nu}p^\mu p^\nu}_{0 \text{ by antisymmetry of } \sigma_{\mu\nu}=\frac{i}{2}[\gamma_\mu,\gamma_\nu]}$$

If there is a parity symmetry then the terms w/ γ_5 are ruled out. Hence,

$$\boxed{\tilde{\Sigma}(p) = a(p^2) + c(p^2)\not{p}.}$$

Note that since $\not{p}^2 = p^2$, $\tilde{\Sigma}(p)$ is a single function of $\not{p}$.

The renormalized fermion propagator is:



$$= \frac{i}{\not p - m + i\epsilon} + \frac{i}{\not p - m + i\epsilon}\left(-i\tilde{\Sigma}(p)\right)\frac{i}{\not p - m + i\epsilon} + \cdots$$

$$= \frac{i}{\not p - m + i\epsilon}\,\frac{1}{1 - \frac{\tilde{\Sigma}(p)}{\not p - m + i\epsilon}}$$

$$= \frac{i}{\not p - m - \tilde{\Sigma}(p) + i\epsilon}$$

What we are calculating is the Fourier transform of $\langle 0 | T(\tilde{\Psi}(x)\bar{\tilde{\Psi}}(0)) | 0 \rangle$.

To determine the renormalization conditions in terms of $\tilde{\Sigma}(p)$ we insert a complete set of states between the fields (vacuum + 1-particle + 2-particle + ...)

$$\langle 0 | T\tilde{\Psi}(x)\bar{\tilde{\Psi}}(0)|0\rangle = \sum_{|n\rangle} \langle 0|\tilde{\Psi}(x)|n\rangle\langle n|\bar{\tilde{\Psi}}(0)|0\rangle$$

$$= \sum_{|n\rangle} e^{-iP_n\cdot x}\langle 0|\tilde{\Psi}(0)|n\rangle\langle n|\bar{\tilde{\Psi}}(0)|0\rangle$$

renormalization condition

$$= \underbrace{\sum_r \int \frac{d^3 q}{(2\pi)^3 2\omega_q} e^{-iq\cdot x}\, u^r(\vec{q})\,\bar{u}^r(\vec{q})}_{\text{Contribution from 1-particle state}} + \text{contribution from } n>1\text{-particle states.}$$

Here $\omega_{\vec{q}} = \sqrt{\vec{q}^{\,2} + m^2}$ where m is the physical mass of the 1-particle states.

The contribution from 1-particle states to the renormalized propagator is of the same form as the free field propagator;

$$\int \langle 0| T\, \tilde{\Psi}(x) \cdot \bar{\tilde{\Psi}}(0)|0\rangle e^{ip\cdot x} d^4x = \frac{i}{\not{p} - m + i\epsilon} + (\text{regular @ } \not{p}=m)$$

Hence, the renormalized propagator has a pole @ the physical mass with residue i. This determines the renormalization conditions in terms of $\tilde{\Sigma}(p)$:

Pole @ $\not{p}=m$ $\longrightarrow$ $\tilde{\Sigma}(m) = 0$

Residue $= i$ $\longrightarrow$ $\left.\frac{d\tilde{\Sigma}}{d\not{p}}\right|_{\not{p}=m} = 0$

It is customary to relate the bare and renormalized Dirac fields as $\tilde{\Psi}(x) \equiv Z_2^{-1/2}\Psi(x)$.

$$\langle 0| T\, \tilde{\Psi}(x) \bar{\tilde{\Psi}}(0)|0\rangle = Z_2^{-1} \langle 0| T\, \Psi(x) \bar{\Psi}(0)|0\rangle$$

$$\frac{i}{\not{p} - m - \tilde{\Sigma}(\not{p}) + i\epsilon} = \frac{i Z_2^{-1}}{\not{p} - m_0 - \Sigma(p) + i\epsilon}$$

Bare mass

Σ 1PI diagrams in $\langle 0| T\, \Psi \bar{\Psi}|0\rangle$

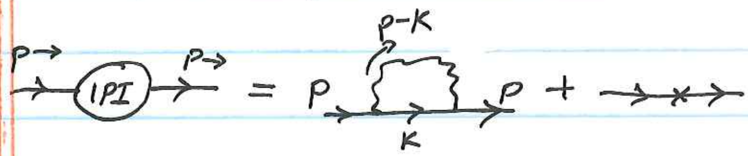
Bare fields

Expanding about $\not{p}=m$, $\boxed{Z_2^{-1} = 1 - \left.\frac{d\Sigma}{d\not{p}}\right|_{\not{p}=m}}$

Electron Self Energy in QED

$$\mathcal{L} = -\frac{1}{4}\tilde{F}_{\mu\nu}\tilde{F}^{\mu\nu} + \bar{\tilde{\Psi}}(i\not{\partial} - e\not{\tilde{A}} - m)\tilde{\Psi} + \mathcal{L}_{CT}$$

$$\mathcal{L}_{CT} = -B\tilde{F}_{\mu\nu}\tilde{F}^{\mu\nu} + C\,\bar{\tilde{\Psi}}\,i\not{\partial}\,\tilde{\Psi} - D\bar{\tilde{\Psi}}\tilde{\Psi} - E\,\bar{\tilde{\Psi}}\not{\tilde{A}}\tilde{\Psi}$$



$+\ \mathcal{O}(e^4)$

$$-i\tilde{\Sigma}(p) = -i\Sigma^{(2)}(p) + iC\not{p} - iD + \mathcal{O}(e^4)$$

Renormalization conditions fix C, D, such that

$$\boxed{\tilde{\Sigma}(p) = \Sigma(p) - \Sigma(m) - \left.\frac{d\Sigma}{d\not{p}}\right|_{\not{p}=m}(\not{p}-m)}$$

Now for the calculation...

2nd order in e

$$-i\Sigma^{(2)}(p) = (-ie)^2\int\frac{d^4k}{(2\pi)^4}\,\gamma^\mu\,\frac{i}{\not{k}-m+i\epsilon}\,\gamma_\mu\,\frac{-i}{(p-k)^2-\mu^2+i\epsilon}$$

We introduce a small "photon mass" μ to regulate the infrared divergence in the integral for $k \approx p$.

We combine denominators using a Feynman parametrization:

$$-i\Sigma^{(2)}(p) = -e^2 \int \frac{d^4k}{(2\pi)^4} \frac{\gamma^\mu(\not{k}+m)\gamma_\mu}{(k^2-m^2+i\epsilon)((p-k)^2-\mu^2+i\epsilon)}$$

$$= -e^2 \int \frac{d^4k}{(2\pi)^4} \int_0^1 dx \frac{\gamma^\mu(\not{k}+m)\gamma_\mu}{\left[((p-k)^2-\mu^2+i\epsilon)x + (k^2-m^2+i\epsilon)(1-x)\right]^2}$$

$$= -e^2 \int_0^1 dx \int \frac{d^4k}{(2\pi)^4} \frac{\gamma^\mu(\not{k}+m)\gamma_\mu}{\left[k^2 - 2k\cdot p x + p^2 x - \mu^2 x - (1-x)m^2 + i\epsilon\right]^2}$$

Complete the square:

$$= -e^2 \int_0^1 dx \int \frac{d^4k}{(2\pi)^4} \frac{\gamma^\mu(\not{k}+m)\gamma_\mu}{\left[(k-px)^2 - (p^2x(x-1)+\mu^2 x + m^2(1-x)) + i\epsilon\right]^2}$$

Shift the momentum: $\ell \equiv k - px$

$$-i\Sigma^{(2)}(p) = -e^2 \int_0^1 dx \int \frac{d^4\ell}{(2\pi)^4} \frac{\gamma^\mu(\not{\ell}+\not{p}x+m)\gamma_\mu}{\left[\ell^2 - (p^2x(x-1)+\mu^2 x + m^2(1-x)) + i\epsilon\right]^2}$$

The term in the integrand $\propto \gamma^\mu \not{\ell} \gamma_\mu$ vanishes upon integration because it is odd under $\ell \to -\ell$.

We can simplify the numerator w/ some γ-matrix algebra:

$$\gamma^\mu \not{p} \gamma_\mu = p_\alpha \gamma^\mu \gamma^\alpha \gamma_\mu = -p_\alpha \gamma^\alpha \gamma^\mu \gamma_\mu + 2p^\mu \gamma_\mu$$
$$= -\not{p}\cdot \tfrac{1}{2} g_{\mu\nu}\{\gamma^\mu,\gamma^\nu\} + 2\not{p}$$
$$= -\not{p}\cdot \tfrac{1}{2} g_{\mu\nu}\cdot 2g^{\mu\nu} + 2\not{p}$$
$$= -2\not{p}$$

$$m\gamma^\mu\gamma_\mu = m\cdot \tfrac{1}{2} g_{\mu\nu}\{\gamma^\mu,\gamma^\nu\} = 4m$$

We can use our integral table for integrals of this form appearing in convergent combinations:

$$I_2(a) \equiv \int \frac{d^4k}{(2\pi)^4} \frac{1}{(k^2+a)^2}$$

$$= \frac{-i}{16\pi^2} \log(-a) + \text{terms that vanish in convergent combinations.}$$

Applying this to our integral:

$$\boxed{-i\Sigma^{(2)}(\not{p}) = -e^2 \int_0^1 dx \left(-2x\not{p} + 4m\right) \frac{(-i)}{16\pi^2} \log\left(p^2 x(x-1) + \mu^2 x + m^2(1-x) - i\epsilon\right)}$$

Renormalized Self Energy to $O(e^2)$:

$$\tilde{\Sigma}^{(2)}(\not{p}) = \Sigma^{(2)}(\not{p}) - \Sigma^{(2)}(m) - \left.\frac{d\Sigma^{(2)}}{d\not{p}}\right|_{\not{p}=m} (\not{p}-m)$$

$$\boxed{\begin{aligned}\tilde{\Sigma}^{(2)}(\not{p}) = \frac{e^2}{(4\pi)^2} \int_0^1 dx \Bigg[&(2x\not{p} - 4m) \log\left(\frac{p^2 x(x-1) + \mu^2 x + m^2(1-x) - i\epsilon}{m^2 x(x-1) + \mu^2 x + m^2(1-x) - i\epsilon}\right) \\ &- \frac{(2xm - 4m)\, 2m x(x-1)\, (\not{p}-m)}{m^2 x(x-1) + \mu^2 x + m^2(1-x) - i\epsilon} \Bigg]\end{aligned}}$$

<u>Comment</u>: $\Sigma^{(2)}(p)$ depends logarithmically on the cutoff Λ. This implies that the <u>bare electron mass</u> must be tuned to a part in $\log\left(\frac{\Lambda}{m}\right)$, compared with the physical electron mass. This is much less serious a tuning than the $O\left(\frac{m}{\Lambda}\right)^2$ tuning for the scalar field, which is why there is a naturalness puzzle for scalars, but not for fermions.