Quantization of Spontaneously Broken Gauge Theories

In unitary gauge the degrees of freedom are manifest. The free part of the Lagrangian describing the massive components of the gauge field has the Proca form:

$$\mathcal{L}_0 = -\frac{1}{4}(\partial_\mu A_\nu^a - \partial_\nu A_\mu^a)(\partial^\mu A^{\nu a} - \partial^\nu A^{\mu a}) + \frac{m^2_{ab}}{2} A_\mu^a A^{\mu b}$$

The mass matrix m^2_{ab} is real and symmetric, so it can be diagonalized with an orthogonal matrix. In the diagonalized basis, $\mathcal{L}_0 = -\frac{1}{4}(\partial_\mu A_\nu^a - \partial_\nu A_\mu^a)^2 + \sum_a \frac{m_a^2}{2} A_\mu^a A^{\mu a}$.

The propagator for the massive gauge fields have the form $\langle 0 | T A_\mu^a(x) A_\nu^b(y) | 0 \rangle$

$$= \frac{\int \mathcal{D}A^\alpha \exp(i\int d^4x\, \mathcal{L}_0)\, A_\mu^a(x) A_\nu^b(y)}{\int \mathcal{D}A^\beta \exp(i\int d^4x\, \mathcal{L}_0)}$$

(Exercise)
$$= \int \frac{d^4k}{(2\pi)^4} \frac{-i e^{-ik\cdot(x-y)}}{k^2 - m_a^2 + i\epsilon}\left(g_{\mu\nu} - \frac{k_\mu k_\nu}{m_a^2}\right)\delta^{ab}$$

Hence, the Feynman rule for the propagator in unitary gauge is

$$\boxed{\overset{\mu}{\underset{a}{}} \overset{k\rightarrow}{\sim\sim\sim} \overset{\nu}{\underset{b}{}} \quad \frac{-i}{k^2 - m_a^2 + i\epsilon}\left(g_{\mu\nu} - \frac{k_\mu k_\nu}{m_a^2}\right)\delta^{ab}}$$

This form of the propagator would seem to make integrals over k^μ more divergent than for the scalar field, because of the $k_\mu k_\nu$ term. This puts renormalizability of the theory in greater question. However, there is an average over gauges that makes the large-k^μ behavior more well-behaved.

R_ξ gauges

Consider the Abelian Higgs model:

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + (D_\mu \phi)^\dagger D^\mu \phi + \mu^2 \phi^\dagger \phi - \lambda (\phi^\dagger \phi)^2$$

Expanding about the VEV $\langle \phi \rangle = v/\sqrt{2} = \frac{1}{\sqrt{2}} \left(\frac{\mu^2}{\lambda}\right)^{1/2}$,

$\mathcal{L} \supset + g v A^\mu \partial_\mu \phi_2'$, where $\phi = \frac{1}{\sqrt{2}} (v + \phi_1' + i \phi_2')$.

We can eliminate the term mixing the gauge field and the would-be Goldstone boson by averaging over gauges w/ gauge-fixing condition

$$\boxed{G[A_\mu, \phi] = \partial^\mu A_\mu + \xi m_A \phi_2 - f(x) = 0}$$

$m_A = g v$.

Temporarily disregarding the Fadeev-Popov determinant (which gives rise to ghosts), we insert into the functional integral:

$$\int \mathcal{D} f(x) \exp\left(-\frac{i}{2\xi} \int f(x)^2 d^4x\right) \delta\left(G[A_\mu, \phi]\right)$$

$$= \exp\left(\frac{-i}{2\xi} \int d^4x \left(\partial^\mu A_\mu + \xi m_A \phi_2\right)^2\right) \equiv \exp\left(i \int d^4x \, \mathcal{L}_{g.f.}\right)$$

where the gauge-fixing Lagrangian has the form

$$\boxed{\mathcal{L}_{g.f.} = -\frac{1}{2\xi} \left(\partial^\mu A_\mu + \xi m_A \phi_2\right)^2} \leftarrow R_\xi \text{ gauge}$$

The free part of the Lagrangian, including $\mathcal{L}_{g.f.}$, is

$$\boxed{\begin{aligned}\mathcal{L}_0 = \frac{1}{2}\left((\partial^\mu \phi_1')^2 - 2\mu^2 \phi_1'^2\right) + \frac{1}{2}\left((\partial^\mu \phi_2')^2 - \xi m_A^2 \phi_2'^2\right) \\ - \frac{1}{4}(\partial_\mu A_\nu - \partial_\nu A_\mu)^2 + \frac{1}{2} m_A^2 A_\mu A^\mu - \frac{1}{2\xi}(\partial^\mu A_\mu)^2\end{aligned}}$$

Note that the mixing between A_μ and ϕ_2' has been eliminated, but unlike in unitary gauge the field ϕ_2' propagates.

Consider the gauge field part of the action:

$$S = \int d^4x \, \frac{1}{2} A_\mu \left(g^{\mu\nu}\partial^2 - \partial^\mu \partial^\nu + m_A^2 g^{\mu\nu} + \frac{1}{\xi}\partial^\mu\partial^\nu\right) A_\nu$$

$$= \int d^4x \, \frac{1}{2} A_\mu \left[(\partial^2 + m_A^2)\left(g^{\mu\nu} - \frac{\partial^\mu\partial^\nu}{\partial^2}\right) + \left(\frac{1}{\xi}\partial^2 + m_A^2\right)\left(\frac{\partial^\mu\partial^\nu}{\partial^2}\right)\right] A_\nu$$

The Kernel of the Gaussian functional integral is

$$K^{\mu\nu}(x,x') = -i\,\delta^4(x-x')\left[(\partial^2 + m_A^2)\left(g^{\mu\nu} - \frac{\partial^\mu\partial^\nu}{\partial^2}\right) + \left(\frac{1}{\xi}\partial^2 + m_A^2\right)\left(\frac{\partial^\mu\partial^\nu}{\partial^2}\right)\right]$$

We can invert the transverse and longitudinal parts separately:

$$K^{-1}_{\nu\alpha}(x',x'') = \int \frac{d^4k}{(2\pi)^4}\left[\frac{-i}{k^2 - m_A^2 + i\epsilon}(P_T)_{\nu\alpha} - \frac{i\,\xi}{k^2 - \xi m_A^2 + i\epsilon}(P_L)_{\nu\alpha}\right] e^{-ik\cdot(x'-x'')}$$

where $(P_T)_{\nu\alpha} = g_{\nu\alpha} - \frac{k_\nu k_\alpha}{k^2}$, $(P_L)_{\nu\alpha} = \frac{k_\nu k_\alpha}{k^2}$.

$$K^{-1}_{\nu\alpha}(x',x'') = \int \frac{d^4k}{(2\pi)^4}\left[\frac{-i}{k^2-m_A^2+i\epsilon}\left(g_{\nu\alpha}-\frac{k_\nu k_\alpha}{k^2}\right) - \frac{i\xi}{k^2-\xi m_A^2+i\epsilon}\frac{k_\nu k_\alpha}{k^2}\right]e^{-ik\cdot(x'-x'')}$$

$$= \int \frac{d^4k}{(2\pi)^4}\left[\frac{-i}{k^2-m_A^2+i\epsilon}\left(g_{\nu\alpha} - \frac{k_\nu k_\alpha}{k^2}\left(1-\frac{\xi(k^2-m_A^2)}{k^2-\xi m_A^2+i\epsilon}\right)\right)\right]$$

$$= \int \frac{d^4k}{(2\pi)^4}\left[\frac{-i}{k^2-m_A^2+i\epsilon}\left(g_{\nu\alpha} - (1-\xi)\frac{k_\nu k_\alpha}{k^2-\xi m_A^2+i\epsilon}\right)\right]$$

<u>Exercise</u>: Check $\int d^4x' \, K^{\mu\nu}(x,x')\, K^{-1}_{\nu\alpha}(x',x'') = \delta^4(x-x'')\,\delta^\mu_\alpha$

From $K^{-1}_{\nu\alpha}$ we read off the Feynman rule for the massive gauge boson in R_ξ gauge:

$$\boxed{\mu \sim\!\sim\!\sim \nu \qquad \frac{-i}{k^2-m_A^2+i\epsilon}\left(g_{\nu\alpha} - (1-\xi)\frac{k_\nu k_\alpha}{k^2-\xi m_A^2+i\epsilon}\right)}$$

☆ Note that for large k_ν, the propagator scales like the scalar field propagator for any finite ξ.

In R_ξ gauge the would-be Goldstone boson also propagates:

$$\mathcal{L}_0 \supset \frac{1}{2}\left((\partial_\mu \phi_2')^2 - \xi m_A^2 \phi_2'^2\right)$$

$$\boxed{\phi_2' \xrightarrow{k} \phi_2' \qquad \frac{i}{k^2-\xi m_A^2+i\epsilon}}$$

☆ As $\xi \to \infty$, ϕ_2' decouples and the gauge field propagator approaches the unitary gauge propagator.