

Ward Identities and Anomalies, Cheng+Li 6.1, 6.2

Ward identities are the consequence of symmetries in quantum field theory.

Consider the theory of a complex scalar field ϕ with a $U(1)$ symmetry $\phi \rightarrow e^{i\theta} \phi$.

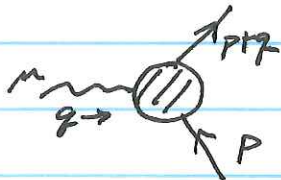
As a consequence there is a conserved current J^μ such that $\partial_\mu J^\mu = 0$. If $\mathcal{L} = (\partial_\mu \phi)^\dagger (\partial^\mu \phi) + f(\phi + \phi^\dagger)$, then $J_\mu = i [(\partial_\mu \phi)^\dagger \phi - (\partial_\mu \phi) \phi^\dagger]$.

Using the equal-time commutator $[\partial_0 \phi^\dagger(\vec{x}, t), \phi(\vec{x}', t)] = -i \delta^3(\vec{x} - \vec{x}')$ we also have the commutators

$$\begin{aligned} [J_0(\vec{x}, t), \phi(\vec{x}', t)] &= i [(\partial_0 \phi)^\dagger(\vec{x}, t), \phi(\vec{x}', t)] \phi(\vec{x}, t) \\ &= \delta^3(\vec{x} - \vec{x}') \phi(\vec{x}, t) \end{aligned}$$

$$[J_0(\vec{x}, t), \phi^\dagger(\vec{x}', t)] = -\delta^3(\vec{x} - \vec{x}') \phi^\dagger(\vec{x}, t)$$

Consider the 3-pt function



$$\Gamma_\mu(p_1, p) = \int d^4x d^4z e^{-i(q \cdot x + p \cdot z)} \langle 0 | T(J_\mu(x) \phi(z) \phi^\dagger(0)) | 0 \rangle$$

$$q^\mu \Gamma_\mu(p_1, p) = -i \int d^4x d^4z e^{-i(q \cdot x + p \cdot z)} \partial_x^\mu \langle 0 | T(J_\mu(x) \phi(z) \phi^\dagger(0)) | 0 \rangle$$

$$= -i \int d^4x d^4z e^{-i(q \cdot x + p \cdot z)} \left[\langle 0 | T(\partial^\mu J_\mu(x) \phi(z) \phi^\dagger(0)) | 0 \rangle \right.$$

$$\left. + \delta(x^0 - z^0) \langle 0 | T([J_0(x), \phi(z)] \phi^\dagger(0)) | 0 \rangle + \delta(x_0) \langle 0 | T([J_0(x), \phi^\dagger(0)] \phi(z)) | 0 \rangle \right]$$

$$\boxed{q^\mu T_m(p, q) = -i \int d^4x e^{-i(p+q)\cdot x} \langle 0 | T[\phi(x) \phi^\dagger(0)] | 0 \rangle$$

$$+ i \int d^4z e^{-i p \cdot z} \langle 0 | T[\phi^\dagger(0) \phi(z)] | 0 \rangle$$

$$\boxed{= -i (\Delta(p+q) - \Delta(p))} \quad \text{Ward identity}$$

where $\Delta(p)$ is the propagator $\Delta(p) = \int d^4x e^{-i p \cdot x} \langle 0 | T[\phi(x) \phi^\dagger(0)] | 0 \rangle$

Now consider a theory of fermions. Define the vector, axial vector, and pseudoscalar currents:

$$V_m(x) = \bar{\psi}(x) \gamma_m \psi(x)$$

$$A_m(x) = \bar{\psi}(x) \gamma_m \gamma_5 \psi(x)$$

$$P(x) = \bar{\psi}(x) \gamma_5 \psi(x)$$

If $\mathcal{L} = \bar{\psi}(i\partial - m)\psi$, then from the equations of motion

$$\partial^\mu V_\mu = 0$$

$$\partial^\mu A_\mu = 2imP$$

$$\text{Use } \partial_x^\mu T(J_\mu(x) \theta(z)) = T(\partial^\mu J_\mu(x) \theta(z)) + [J_0(x), \theta(z)] \delta(x^0 - z^0)$$

$$\text{and } [V_0(x), A_0(z)] \delta(x^0 - z^0) = 0$$

$$\text{Define } T_{\mu\nu\lambda}(k_1, k_2, q) = i \int d^4x d^4z \langle 0 | T(V_\mu(x) V_\nu(z) A_\lambda(0)) | 0 \rangle e^{i(k_1 \cdot x + k_2 \cdot z)}$$

$$T_{\mu\nu}(k_1, k_2, q) = i \int d^4x d^4z \langle 0 | T(V_\mu(x) V_\nu(z) P(0)) | 0 \rangle e^{i(k_1 \cdot x + k_2 \cdot z)}$$

Exercise.

\Rightarrow

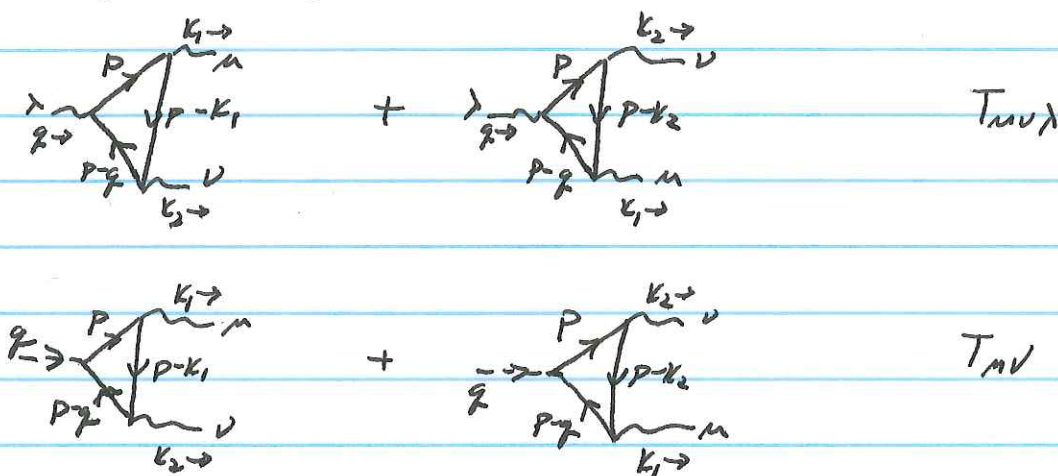
$$k_1^\mu T_{\mu\nu\lambda} = k_2^\nu T_{\mu\nu\lambda} = 0$$

$$q^\lambda T_{\mu\nu\lambda} = 2m T_{\mu\nu}$$

Ward identities.

We assumed the classical current conservation relations in the derivation of the Ward identities. In chiral theories renormalization may violate these relations.

Triangle Anomaly:



$$T_{\mu\nu\lambda} = i \int \frac{d^4 p}{(2\pi)^4} (-1) \text{Tr} \left[\frac{i}{\not{p} - m + i\epsilon} \gamma_\mu \gamma_5 \frac{i}{\not{p} - \not{k}_2 - m + i\epsilon} \gamma_\nu \frac{i}{\not{p} - \not{k}_1 - m + i\epsilon} \gamma_\lambda \right] + \left(\begin{matrix} k_1 \leftrightarrow k_2 \\ \mu \leftrightarrow \nu \end{matrix} \right)$$

$$T_{\mu\nu} = i \int \frac{d^4 p}{(2\pi)^4} (-1) \text{Tr} \left[\frac{i}{\not{p} - m + i\epsilon} \gamma_5 \frac{i}{\not{p} - \not{q} - m + i\epsilon} \gamma_\nu \frac{i}{\not{p} - \not{k}_1 - m + i\epsilon} \gamma_\mu \right] + \left(\begin{matrix} k_1 \leftrightarrow k_2 \\ \mu \leftrightarrow \nu \end{matrix} \right)$$

Use $\not{q} \gamma_5 = \gamma_5 (\not{p} - \not{q} - m) + (\not{p} - m) \gamma_5 + 2m \gamma_5$

$$\rightarrow q^\lambda T_{\mu\nu\lambda} = 2m T_{\mu\nu} + \Delta_{\mu\nu}^{(1)} + \Delta_{\mu\nu}^{(2)}$$

$$\Delta_{\mu\nu}^{(1)} = \left(\frac{d^4 p}{(2\pi)^4} \right) \text{Tr} \left[\frac{i}{\not{p} - m + i\epsilon} \delta_5 \delta_\nu \frac{i}{(\not{p} - k_1) - m + i\epsilon} \delta_\mu \right. \\ \left. - \frac{i}{(\not{p} - k_2) - m + i\epsilon} \delta_5 \delta_\nu \frac{i}{(\not{p} - k_1) - m + i\epsilon} \delta_\mu \right]$$

$$\Delta_{\mu\nu}^{(2)} = \left(\frac{d^4 p}{(2\pi)^4} \right) \text{Tr} \left[\frac{i}{\not{p} - m + i\epsilon} \delta_5 \delta_\mu \frac{i}{(\not{p} - k_2) - m + i\epsilon} \delta_\nu \right. \\ \left. - \frac{i}{(\not{p} - k_1) - m + i\epsilon} \delta_5 \delta_\mu \frac{i}{(\not{p} - k_2) - m + i\epsilon} \delta_\nu \right]$$

The Ward identity is satisfied if $\Delta_{\mu\nu}^{(1)} + \Delta_{\mu\nu}^{(2)} = 0$.

Shifting the integral in the second term of $\Delta_{\mu\nu}^{(1)}$, $p \rightarrow p + k_2$, then using $k_2 \cdot \gamma = k_1 \cdot \gamma$, the two integrals in $\Delta_{\mu\nu}^{(1)}$ would cancel.

Similarly the two integrals in $\Delta_{\mu\nu}^{(2)}$ would cancel shifting $p \rightarrow p + k_1$ in the second integral.

★ However, these integrals are linearly divergent and we cannot shift the integration variable without changing the integral. As a result we will find $\Delta_{\mu\nu}^{(1)} + \Delta_{\mu\nu}^{(2)} \neq 0$, and the Ward identity is not satisfied.

Linearly divergent integrals: $\Delta(a) \equiv \int_{-\infty}^{\infty} dx [f(x+a) - f(x)]$

$$\text{Expand: } \Delta(a) = \int_{-\infty}^{\infty} dx \left[a f'(x) + \frac{a^2}{2} f''(x) + \dots \right]$$

$$= a [f(\infty) - f(-\infty)] + \frac{a^2}{2} [f'(\infty) - f'(-\infty)] + \dots$$

If $\int_{-\infty}^{\infty} f(x) dx$ converges or diverges at most logarithmically, then $0 = f(\pm\infty) = f'(\pm\infty) = \dots \rightarrow \Delta(a) = 0$.

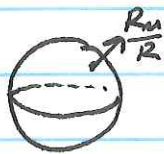
For a linearly divergent integral, $0 \neq f(\pm\infty)$
 $0 = f'(\pm\infty) = \dots$

$$\rightarrow \Delta(a) = a [f(\infty) - f(-\infty)]$$

In n dimensions, $\Delta(a) \equiv \int d^n r [f(r+a) - f(r)]$

$$\Delta(a) = \int d^n r \left[a^m \partial_m f(r) + \frac{1}{2} a^m \partial_m a^{\nu} \partial_{\nu} f(r) + \dots \right]$$

$$= a^m \frac{R_m}{R} f(R) S_n(R)$$



↑ surface area of n -dim' hyper-sphere
 w/ radius R .

Minkowski $n=4$: $\Delta(a) = a^m \int d^4 r \partial_m f(r) = 2i\pi^2 a^m \lim_{R \rightarrow \infty} R^2 R_m f(R)$

Our calculation of $T_{\mu\nu}$ is ambiguous because it depends on the definition of the loop momenta.

It is not invariant under shifts of the momentum integration variable.

We can take $p \rightarrow p + \underbrace{\alpha k_1 + (\alpha - \beta) k_2}_a$

$$\Delta_{\mu\nu\lambda}(a) \equiv T_{\mu\nu\lambda}(a) - T_{\mu\nu\lambda}(0)$$

$$= (-1) \int \frac{d^4 p}{(2\pi)^4} \left(\text{Tr} \left[\frac{1}{\cancel{p} + \cancel{a} - m + i\epsilon} \gamma_\lambda \gamma_5 \frac{1}{\cancel{p} + \cancel{a} + \cancel{a} - m + i\epsilon} \cancel{\sigma}_\nu \frac{1}{(\cancel{p} + \cancel{a} + \cancel{a}) - m + i\epsilon} \gamma_\mu \right] \right. \\ \left. - \text{Tr} \left[\frac{1}{\cancel{p} - m + i\epsilon} \gamma_\lambda \gamma_5 \frac{1}{\cancel{p} + \cancel{a} - m + i\epsilon} \cancel{\sigma}_\nu \frac{1}{\cancel{p} - \cancel{a} - m + i\epsilon} \gamma_\mu \right] \right)$$

$$+ \begin{pmatrix} k_1 \leftrightarrow k_2 \\ \mu \leftrightarrow \nu \end{pmatrix}$$

$$\equiv \Delta_{\mu\nu\lambda}^{(1)} + \Delta_{\mu\nu\lambda}^{(2)}$$

$$\Delta_{\mu\nu\lambda}^{(1)} = (-1) \int \frac{d^4 p}{(2\pi)^4} a^\sigma \frac{\partial}{\partial p^\sigma} \text{Tr} \left[\frac{1}{\cancel{p} - m + i\epsilon} \gamma_\lambda \gamma_5 \frac{1}{\cancel{p} - m + i\epsilon} \cancel{\sigma}_\nu \frac{1}{\cancel{p} - \cancel{a} - m + i\epsilon} \gamma_\mu \right]$$

$$= \frac{-2i\pi^2 a^\sigma}{(2\pi)^4} \lim_{p \rightarrow \infty} p^\sigma p_\sigma \text{Tr} \left[\frac{\cancel{p} \gamma_\lambda \gamma_5 \cancel{p} \cancel{\sigma}_\nu \cancel{p} \gamma_\mu}{p^6} \right]$$

$$= \frac{2i\pi^2 a^\sigma}{(2\pi)^4} \lim_{p \rightarrow \infty} \frac{p_\sigma p^\sigma}{p^2} 4i \epsilon_{\mu\nu\lambda\rho}$$

Replace $p^\sigma p^\rho \rightarrow p^2 g^{\rho\sigma}/4$

$$\boxed{\Delta_{\mu\nu\lambda}^{(1)} = \epsilon_{\rho\mu\nu\lambda} \frac{a^\rho}{8\pi^2}}$$

$\Delta_{\mu\nu\lambda}^{(2)}$ is the same as $\Delta_{\mu\nu\lambda}^{(1)}$ with the exchanges
 $k_1 \leftrightarrow k_2, \mu \leftrightarrow \nu$.

Using $q = \alpha k_1 + (\alpha - \beta) k_2$, we get

$$\Delta_{\mu\nu\lambda} = \Delta_{\mu\nu\lambda}^{(1)} + \Delta_{\mu\nu\lambda}^{(2)} = \frac{\beta}{8\pi^2} \epsilon_{\rho\mu\nu\lambda} (k_1 - k_2)^\rho$$

Hence $T_{\mu\nu\lambda}$ is ambiguous up to the addition

$$T_{\mu\nu\lambda} \rightarrow T_{\mu\nu\lambda} - \frac{\beta}{8\pi^2} \epsilon_{\mu\nu\lambda\rho} (k_1 - k_2)^\rho$$

Similarly, $\Delta_{\mu\nu}^{(1)} = \Delta_{\mu\nu}^{(2)} = -\frac{1}{8\pi^2} \epsilon_{\mu\nu\sigma\rho} k_1^\sigma k_2^\rho$

$$\Rightarrow q^\lambda T_{\mu\nu\lambda} = 2m T_{\mu\nu} - \frac{(1-\beta)}{4\pi^2} \epsilon_{\mu\nu\sigma\rho} k_1^\sigma k_2^\rho$$

where β is so far undetermined

Similarly, $k_1^\mu T_{\mu\nu\lambda} = \frac{(1+\beta)}{8\pi^2} \epsilon_{\nu\lambda\sigma\rho} k_1^\sigma k_2^\rho$ (Exercise)

The terms $\alpha(1-\beta)$ and $(1+\beta)$ are the anomalous terms.
 It is not possible to simultaneously set both of these terms to zero.

To maintain the vector Ward identity we choose

$$\boxed{\beta = -1}$$

$$\Rightarrow q^\lambda T_{\mu\nu\lambda} = 2m T_{\mu\nu} - \frac{1}{2\pi^2} \epsilon_{\mu\nu\sigma\rho} k_1^\sigma k_2^\rho$$

The anomalous term corresponds to a modification of the classical current divergence equation:

$$\partial^\mu A_\mu = 2 \text{im } P + \underbrace{\frac{1}{(4\pi)^2} \epsilon^{\mu\nu\rho\sigma} F_{\mu\nu} F_{\rho\sigma}}_{\text{The ABJ anomaly}}$$

Field strengths for
gauge vector symmetry
~ electromagnetism.

The ABJ anomaly

^ Adler, Bell, Jackiw (1969)

Non-Abelian anomalies:

$$T_{\mu\nu}^{abc}(k_1, k_2, q) = i \int d^4x d^4y \langle 0 | T(V_\mu^a(x) V_\nu^b(y) A_\lambda^c(0)) | 0 \rangle e^{i(k_1 x + k_2 y)}$$

where

$$V_\mu^a = \bar{\psi} T^a \gamma_\mu \psi$$

$$A_\lambda^c = \bar{\psi} T^c \gamma_\lambda \gamma_5 \psi$$

Also define

$$P^c = \bar{\psi} T^c \gamma_5 \psi$$

$$\text{Then } q^\lambda T_{\mu\nu}^{abc} = 2 \text{im } T_{\mu\nu}^{abc} + \text{commutators} - \frac{1}{2\pi^2} \epsilon^{\mu\nu\rho\sigma} k_1^\rho k_2^\sigma D^{abc}$$

ABJ Anomaly

where $D^{abc} = \frac{1}{2} \text{Tr}(\{T^a, T^b\} T^c)$

Note that left-handed spinors contribute to the anomaly with the opposite sign as right-handed spinors in the same representation.

Gauge anomalies must vanish for renormalizability.

Hence, if T^a, T^b, T^c are generators of gauge invariances then summed over representations of chiral fermions, $D^{abc} = 0$.

For $U(1)$ gauge invariance, we need $\sum Q_i^3 = 0$ where $Q_i = \text{fermion charge}$.

To couple to gravity, we can replace two T^a 's w/ $T_{\mu\nu}$ stress tensor $\rightarrow \sum Q_i = 0$