The Renormalization Group (RG)

Stueckelberg and Peterman (1953) – first introduced

Gell-Mann, Low (1954) – applied to asymptotic behavior of QED

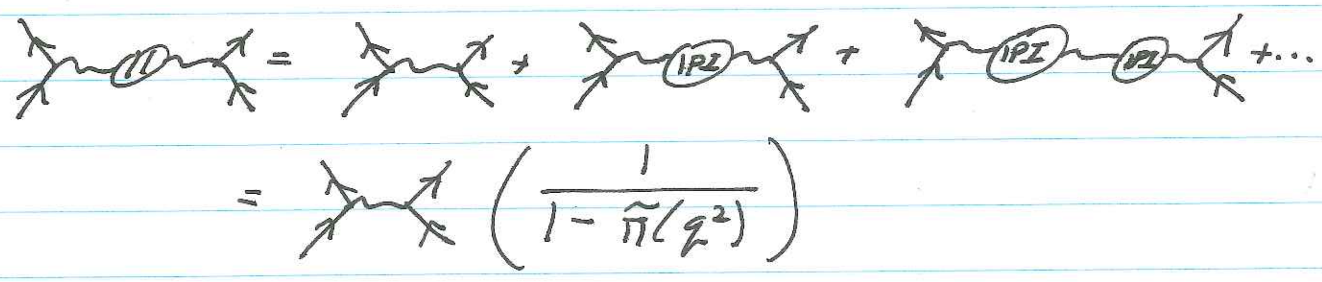
Bogoliubov, Shirkov (1959) – clarification

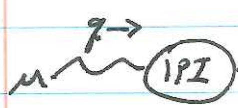
Kadanoff (1966), Wilson (1969) – physical picture of RG

Callan, Symanzik (1970) – RG eqn by momentum subtraction schemes

<u>Lesson from QED</u>: Screening of electric charge by vacuum polarization – depends on momenta involved in scattering experiment

Electron-electron scattering in QED:





$= i\left(q^2 g_{\mu\nu} - q_\mu q_\nu\right)\tilde{\Pi}(q^2)$

Renormalization Condition: $\tilde{\Pi}(0) = 0$

$\rightarrow$ fixes residue of the pole in the photon propagator 

The effect of renormalization of the photon propagator is to replace

$$e^2 \to \boxed{\frac{e^2}{1-\tilde{\Pi}(q^2)} \equiv e^2_{eff}(q^2)}$$

The 1-loop contribution to the photon self energy is straightforward to calculate

counterterm from
$\mathcal{L} \supset -\frac{A}{4} F_{\mu\nu}F^{\mu\nu}$
shifts $\tilde{\Pi}(q^2)$ by const.

$$\Rightarrow \tilde{\Pi}(q^2) = \frac{e^2}{2\pi^2}\int_0^1 dx\, x(1-x)\log\left(\frac{m^2 - x(1-x)q^2}{m^2}\right)$$

$$\approx \frac{e^2}{12\pi^2}\left[\log\left(\frac{-q^2}{m^2}\right) - \frac{5}{3} + \mathcal{O}\left(\frac{m^2}{q^2}\right)\right]$$

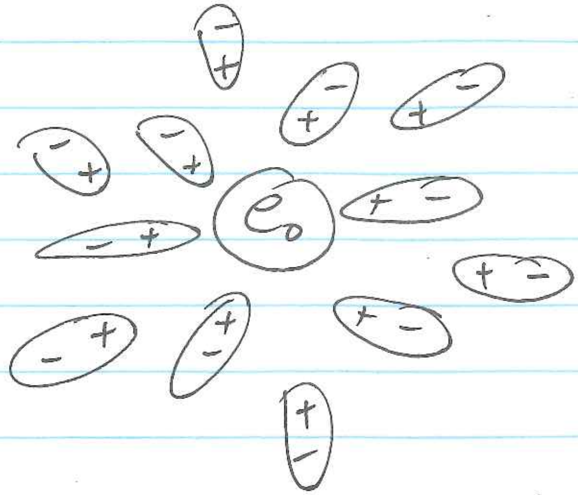
(for large $\frac{-q^2}{m^2}$).

$$\longrightarrow \boxed{e^2_{eff}(q^2) \approx \frac{e^2}{1 - \frac{e^2}{12\pi^2}\log\left(\frac{-q^2}{e^{5/3}m^2}\right)}}$$

<u>Running Coupling</u>

The effective electric charge <u>increases at shorter distances.</u>

Physical interpretation: Screening by vacuum polarization



Heuristic picture:
Virtual electron-positron pairs turn the environment of the bare electron into a kind of dielectric, which is polarized by the presence of the bare charge.

The effective charge measured at some momentum scale $e_{eff}^2(q^2)$ is smaller than e_0^2 because of screening. At higher momentum (shorter distance) more of the bare charge is seen, so the effective charge grows in magnitude.

The most important information in $e_{eff}^2(q^2)$ is the factor $\frac{e^2}{12\pi^2}$ multiplying the log in the denominator. That factor determines how the effective charge changes as the momentum is rescaled. Hence, it is natural to define:

convention

$$\boxed{\beta(e_{eff}) = -e_{eff}^3 \frac{d\left(\frac{1}{e_{eff}^2}\right)}{d\log\left(\frac{-q^2}{m^2}\right)}}$$

β-function

$$\boxed{\beta_{QED}(e_{eff}) \approx \frac{e_{eff}^3}{12\pi^2}}$$

The renormalization group equation inverts this logic. If we are handed a $\beta(e_{eff})$, then we can solve the RG equation for $e_{eff}(q^2)$ to determine how the effective charge (coupling) varies with momentum scale. A simple manipulation of the equation defining $\beta(e_{eff})$ gives the RG equation:

$$\frac{\partial e_{eff}(q, e)}{\partial \log\left(\frac{-q^2}{m^2}\right)} = \frac{\beta}{2}$$

In terms of the electric potential (Fourier transformed),

$$V(q, e) \approx \frac{e_{eff}^2(q, e)}{q^2}$$

we can write

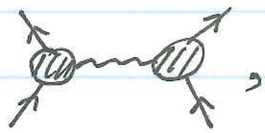
$$\boxed{\left[q\frac{\partial}{\partial q} - \beta(e_{eff})\frac{\partial}{\partial e_{eff}} + 2\right]V = 0}$$

This is an example of a <u>Callan-Symanzik eqn</u>, a class of equations which describes how Green's functions evolve as momentum scale is varied. One of our goals is to generalize this.

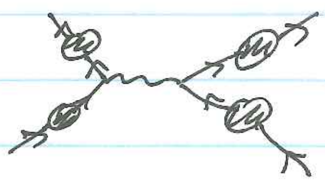
The β-function in non-Abelian Gauge Theories:

The β-fn determines how the renormalized coupling varies w/ energy scale. In QED the photon self-energy completely determined the β-function.

Other diagrams that contribute to electron scattering are vertex corrections,



and electron self energy corrections:



, These electron self energy corrections do not contribute to the amplitude (ie they are amputated) if we understand the external electrons to be on-shell.

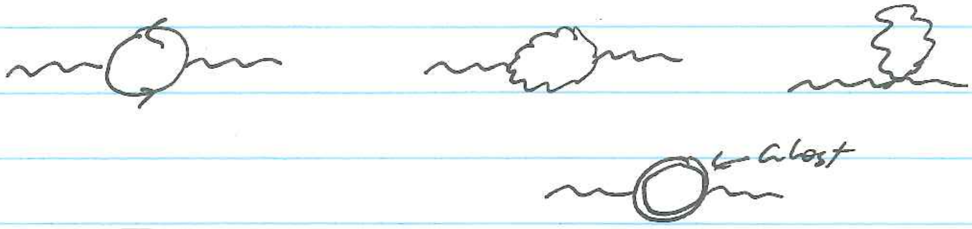
The self energy corrections account for the difference between the bare and renormalized masses, and the wavefunction renormalization of the electron.

If we consider inserting the vertex somewhere inside a Feynman diagram, then the electron self-energy corrections will be important. To keep track of this, rather than study the RG running of the electric potential, which in the non-Abelian case must carefully be made gauge invariant, we study the Green's fn.

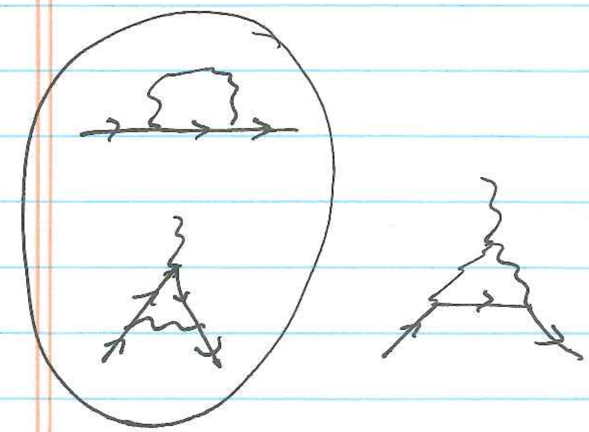
$\langle 0|T(\bar{\Psi}(p_1)\Psi(p_2)A^{aT}_{\mu}(q))|0\rangle$, where the T = transverse part.

All 3 types of corrections appear in calculation of the renormalized Green's fn: Ψ wavefn renorm, A^a_μ wavefn renorm, vertex corrections.

The relevant Feynman diagrams are



Gauge Boson Self Energy



Fermion Self Energy

Vertex corrections

The contributions of the circled diagrams to the β-function cancel. In QED that's why only the photon self energy is important. The additional diagram contributing to the vertex correction is absent in QED.

We will return to the meaning of the β-fn in terms of the generalized Callan-Symanzik eqn, but for now we will just keep in mind that it tells us how couplings run with energy scale.

Without derivation, we quote the one-loop β-function for non-Abelian gauge theories:

or $\times\frac{1}{2}$ for real scalars

$$\boxed{\beta(g) = \frac{-g^3}{(4\pi)^2}\left[\frac{11}{3}C_2(G) - \frac{4}{3}\sum_{\substack{\text{fermions}\\ \text{reps}}}\mu_f - \frac{1}{3}\sum_{\substack{\text{scalars}\\ \text{reps}}}\mu_s\right]}$$

or $\times\frac{1}{2}$ for chiral fermions

$C_2(G)$ is the quadratic Casimir of the gauge group: If T^a_{adj} are the generators of the gauge group in the adjoint rep, then

$$\boxed{\sum_a T^a_{adj}T^a_{adj} = C_2(G)\,\mathbb{1}.}$$

Also, $Tr\left(T^a_{adj}T^b_{adj}\right) = C_2(G)\,\delta^{ab} \equiv \mu(G)\,\delta^{ab}$

μ_f is the Dynkin index of the representation of the gauge group under which a set of fermions transform,

Peskin & Schroeder call this $C(rep)$

$$\boxed{Tr\, T^a_{rep}T^b_{rep} = \mu_{rep}\,\delta^{ab}}$$

Similarly, μ_s is the Dynkin index of the representation of the gauge group under which a set of scalars transform

If the fundamental rep of $SU(N)$ is normalized s.t.

$\boxed{\mu_{fund} = \frac{1}{2}}$, then $\boxed{C_2(SU(N)) = N}$

Rescaling μ_{fund} also rescales the coupling g.

For QED, $C_2(G) \to 0$ (those Feynman diagrams w/ multi-photon vertices don't exist)

$N_f \to 1$ one electron

$N_s \to 0$ no scalars

$$\beta(e)_{(1\text{-loop})} = \frac{-e^3}{(4\pi)^2}\left(-\frac{4}{3}\right) = \frac{e^3}{12\pi^2} \text{, as before.}$$

For QCD w/ N_f flavors of quarks ($N_f = 6$ in the Standard Model)

$$\beta(g)_{(1\text{-loop})} = \frac{-g^3}{(4\pi)^2}\left[\frac{11}{3}\cdot 3 - \frac{4}{3}\cdot\frac{1}{2}N_f\right]$$

$$\boxed{\beta(g) = \frac{-g^3}{(4\pi)^2}\left[11 - \frac{2}{3}N_f\right]}$$

$$= \frac{g^3}{(4\pi)^2}\cdot 7 \quad \text{if } N_f = 6$$

As long as $N_f < 17$, $\beta(g)_{(1\text{-loop})} < 0$.

This is the opposite sign to the QED β-function, and leads to <u>asymptotic freedom</u>.

Renormalization Group Running

Consider the RG eqn,

$$\frac{\partial g}{\partial \log(\frac{-q^2}{m^2})} = \frac{1}{2}\beta(g)$$

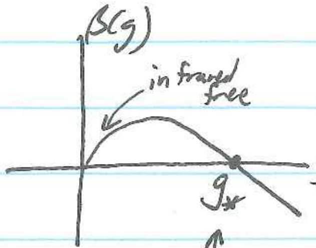
Assuming $g \ll 1$, at least for some range of the couplings, we can analyze the structure of the sol'ns to the RG eqn.

1) $\beta(g) > 0$
This is like QED. The coupling grows at high energies, decreases at low energies.

2) $\beta(g) < 0$
This is asymptotic freedom, like QCD. The coupling is small at high energies, strong at low energies.

3) $\beta(g) = 0$
This is a special case. The coupling does not change as the energy is rescaled. If all of the terms in the Lagrangian have this feature, the field theory at that value of g is <u>scale invariant</u>.
$\rightarrow$ Conformal Fixed point if this happens for special coupling g.

Fixed point Couplings are generically of one of two types:



Ultraviolet stable fixed point

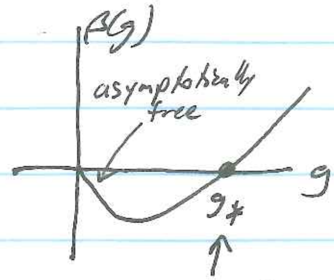
Move a little away from g_*

For $g > g_*$, $\beta(g) < 0$

$\frac{\partial g}{\partial \log(-q^2/m^2)} < 0$

g decreases back towards g_* as $\frac{-q^2}{m^2}$ increases.

Conversely, starting w/ $g < g_*$



Infrared stable fixed pt

For $g > g_*$, $\beta(g) > 0$

$\frac{\partial g}{\partial \log(-q^2/m^2)} > 0$

g decreases back towards g_* as $\frac{-q^2}{m^2}$ decreases.

Conversely, starting w/ $g < g_*$.