We are now in a position to construct a non-Abelian gauge theory from knowledge of the gauge group and the representations under which the matter fields transform.

Example: Quantum Chromodynamics = The Strong Interaction

- SU(3) gauge theory
- Six flavors of quarks (u, d, c, s, t, b), each of which transforms in the fundamental representation of the SU(3) gauge group.

$SU(3) \Rightarrow$ 8 gauge fields A_μ^a, $a = 1, \ldots, 8$
(because $\dim SU(3) = 8$)

Fundamental rep = defining rep = 3-dimensional for SU(3)
$\Rightarrow$ 3 colors of quarks for each flavor.

Label quark fields q_I^i, $I = u, d, c, s, t, b$ — flavor
$i = 1, 2, 3$ — color

Generators of SU(3) T^a, $a = 1, \ldots, 8$

Gauge coupling g_3

The Lagrangian for QCD is then:

$$\mathcal{L}_{QCD} = -\frac{1}{4} F_{\mu\nu}^{a} F^{\mu\nu a} + \sum_{I,i,j} \bar{q}_I^{\,i} \left(i\not{\partial}\, \delta_{ij} - g_3 \not{A}^a T^a_{ij} \right) q_I^j$$

+ quark mass terms.

In the Standard Model quark (and lepton) masses are due to couplings to the Higgs fields, which requires one more ingredient.

Pseudoreality of SU(2) Representations:

A representation of a group is called pseudoreal if there is a nonsingular matrix S such that

$$\boxed{S T^a S^{-1} = -T^{a*}}.$$

In that case a representation is equivalent to its conjugate.

The SO(N) groups have all real representations. Since $SU(2) \sim SO(3)$, the integer-spin reps of SU(2) are real.
The half-odd-integer spin reps are called spinor reps. For them, there is a nontrivial S that relates the rep to its conjugate.

Consider the spin-½ representation:

$$T^1 = \frac{\sigma^1}{2}, \quad T^2 = \frac{\sigma^2}{2}, \quad T^3 = \frac{\sigma^3}{2}$$

$$\sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \sigma^{1*} \Rightarrow \boxed{-T^{1*} = -T^1}$$

$$\sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = -\sigma^{2*} \Rightarrow \boxed{-T^{2*} = +T^2}$$

$$\sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \sigma^{3*} \Rightarrow \boxed{-T^{3*} = -T^3}$$

Let $\boxed{S = -i\sigma^2}$.

$$ST^1S^{-1} = \sigma^2 \frac{\sigma^1}{2} \sigma^2 = -(\sigma^2)^2 \frac{\sigma^1}{2} = -\frac{\sigma^1}{2} = -T_1{}^*$$

$$ST^2S^{-1} = \sigma^2 \frac{\sigma^2}{2} \sigma^2 = \frac{\sigma^2}{2} = -T_2{}^*$$

$$ST^3S^{-1} = \sigma^2 \frac{\sigma^3}{2} \sigma^2 = -\frac{\sigma^3}{2} = -T_3{}^*$$

Hence, $\boxed{ST^aS^{-1} = -T^{a*}}$ for $a=1,2,3$.

Suppose $\Phi = \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}$ transforms as an $SU(2)$ doublet,

$$\Phi \to \exp\left[i\theta^a \frac{\sigma^a}{2}\right] \Phi$$

Then,

$$\Phi^* \to \exp\left[-i\theta^a \frac{(\sigma^a)^T}{2}\right] \Phi^*$$

(*) The combination $\boxed{i\sigma^2 \Phi^* = \begin{pmatrix} \phi_2^* \\ -\phi_1^* \end{pmatrix}}$ transforms as an $SU(2)$ doublet, as we will now check.

$$i\sigma^2 \Phi^* \to i\sigma^2 \exp\left[-i\theta^a \frac{(\sigma^a)^T}{2}\right] \Phi^*$$
$$= i\sigma^2 \sum_{n=0}^{\infty} \frac{1}{n!} \left[-i\theta^a \frac{(\sigma^a)^T}{2}\right]^n \Phi^*$$

Expand $\left[-i\theta^a \frac{(\sigma^a)^T}{2}\right]^n$ and insert $(-i\sigma^2)(i\sigma^2) = 1$ between factors of $(\sigma^a)^T$, and to the right of the last factor.

Then, $i\sigma^2 \Phi^* \to \sum_n \frac{1}{n!}\left[i\theta^a \frac{\sigma^a}{2}\right]^n i\sigma^2 \Phi^*$

$$= \exp\left[i\theta^a \frac{\sigma^a}{2}\right]\left(i\sigma^2 \Phi^*\right)$$

This is what we wanted to prove:
Φ and $(i\sigma^2 \Phi^*)$ transform the same way.

The Higgs sector of the Standard Model takes advantage of this fact.

The Higgs fields $\Phi = \begin{pmatrix}\phi_1 \\ \phi_2\end{pmatrix}$ are a pair of complex scalar fields that transform as an $SU(2)_w$ doublet under the weak interactions. — (This is a gauge invariance, not a global symmetry.)

The left-handed fermions (quarks or leptons) form $SU(2)_w$ doublets.

The right-handed fermions are $SU(2)_w$ singlets, i.e. they are invariant under $SU(2)_w$ transformations.

The Lagrangian $\boxed{\mathcal{L}_{Yuk_d} = \lambda_d \left(\bar{\Psi}_L^1, \bar{\Psi}_L^2\right)\begin{pmatrix}\phi_1 \\ \phi_2\end{pmatrix} \psi_R}$ + h.c.

is $SU(2)_w$-invariant. But so is

$$\boxed{\mathcal{L}_{Yuk_u} = \lambda_u \left(\bar{\Psi}_L^1, \bar{\Psi}_L^2\right) i\sigma^2 \begin{pmatrix}\phi_1^* \\ \phi_2^*\end{pmatrix} \psi_R'} + \text{h.c.}$$

Both types of Yukawa couplings are required in the Standard Model for understanding fermion masses.

If the Higgs doublet has a potential $V(\phi_1, \phi_2)$ with a minimum at $\begin{pmatrix}\phi_1\\ \phi_2\end{pmatrix} = \begin{pmatrix}0\\ v\end{pmatrix}$ with v real,

then expanding the Lagrangian about this minimum gives rise to terms of the form

$$\mathcal{L}_d = \lambda_d (\bar{\Psi}_L^1, \bar{\Psi}_L^2)\begin{pmatrix}0\\ v\end{pmatrix}\psi_R \quad + h.c.$$

$$= (\lambda_d v)\bar{\Psi}_L^2 \psi_R + h.c.$$

This looks like a mass term for the Dirac spinor whose left-handed component is Ψ_L^2 and whose right-handed component is ψ_R, with mass $(\lambda_d v)$.

$$\mathcal{L}_u = \lambda_u (\bar{\Psi}_L^1, \bar{\Psi}_L^2)\begin{pmatrix}v\\ 0\end{pmatrix}\psi_R' \quad + h.c.$$

$$= (\lambda_u v)\ \bar{\Psi}_L^1 \psi_R' + h.c.$$

This is a mass term for Ψ_L^1 and ψ_R'.

If it weren't for pseudoreality of $SU(2)$ representations, we couldn't have written $\mathcal{L}_u$ consistent w/ the weak $SU(2)$, and ψ' would be massless.