Renormalizable vs. Nonrenormalizable Field Theories

So far all of the UV divergences we have encountered proved to be nonphysical — there were counterterms available to cancel them. In some theories only a finite number of counterterms in the Lagrangian are necessary in order to cancel divergences. Such theories are called renormalizable.

If UV divergences cannot be cancelled by a finite number of counterterms, a theory is called nonrenormalizable. Nonrenormalizable theories cannot be entirely predictive because an infinite number of experiments would have to be done just to specify the Lagrangian. However, such theories play an important role in the context of effective field theories which are only supposed to be predictive below some energy scale, while the full theory is secretly renormalizable.

It would be nice if we had some simple rules for distinguishing normalizable from nonrenormalizable theories. That will be our current goal. To that end we should study the divergence structure of general quantum field theories.

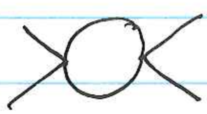
Example: Real scalar field ϕ

$$\mathcal{L} = \cdots - \frac{\lambda}{4!}\phi^4$$

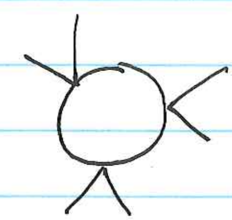
 $\sim \frac{(-ig)^2}{6} \int^{\Lambda} \frac{d^8 q \; i^3}{q^6}$ quadratically divergent ($+ p^2 \log \Lambda$)

Symmetry factor from exchanging 3 internal lines.

The counterterm  cancels the divergence.

 $\sim \frac{(-ig)^2}{2} \int^{\Lambda} \frac{d^4 q}{q^4} i^2$ log divergent

The counterterm  cancels the divergence

 $\sim \frac{(-ig)^3}{1} \int \frac{d^4 q}{q^6}$ finite.

This diagram is finite. No  counterterm is necessary in this theory.

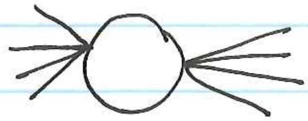
This is a <u>renormalizable</u> theory — only a finite number of counterterms is necessary, so only a finite # of physical definitions for the couplings are necessary.

Example: $\mathcal{L} = \cdots - \frac{g_5}{5!}\phi^5$

 log divergent

$\rightarrow$ Requires ϕ^6 counterterm. 

Then there are diagrams like

 and , log divergent.

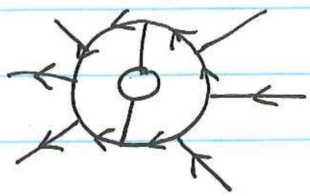
$\rightarrow$ require ϕ^7 and ϕ^8 counterterms.

This pattern continues — an infinite number of counterterms, and physical definitions for couplings, are required in this theory. This is a nonrenormalizable theory.

In the scalar theory it seems like interactions of the form ϕ^p w/ $p \leq 4$ are good for renormalizability, while $p > 4$ is bad.

Let's consider a general Feynman diagram in a general theory of scalars and spin-½ fermions. (Gauge fields are a bit more tricky, but as long as the gauge invariance is maintained by the regulator, gauge fields count as scalars for this purpose.)

Consider some diagram:



$B =$ # external boson lines

$I_B =$ # internal boson lines

$F =$ # external fermion lines

$I_F =$ # internal fermion lines

$n_i =$ # vertices with b_i bosons, f_i fermions, d_i derivatives.

Every internal boson line has 2 ends attached to vertices.
" external " " 1 " " "

Hence, $\boxed{B + 2I_B = \sum n_i b_i}$

Similarly for fermions: $\boxed{F + 2I_F = \sum n_i f_i}$

The <u>superficial degree of divergence</u> of a diagram is the # momentum integrals − # momenta in denominator (net)

$$D = \underbrace{\sum n_i d_i}_{\substack{\text{derivative} \\ \to \text{momentum}}} + \underbrace{2 I_B}_{\substack{\text{internal boson} \\ \to \int \frac{d^4 q}{q^2}}} + \underbrace{3 I_F}_{\substack{\text{internal fermion} \\ \to \int \frac{d^4 q}{\not{q}}}} - \underbrace{4 \sum n_i}_{\substack{\text{Each vertex} \\ \to \delta^4(\sum q_i)}} + \underbrace{4}_{\substack{\text{one } \delta^4(\sum q_i) \\ \text{left over}}}$$

The mass dimension of the operators in the Lagrangian are

$\boxed{\dim \mathcal{L}_i = b_i + \frac{3}{2} f_i + d_i}$ (in the theories we have considered)

$$D = \sum n_i d_i + 2 I_B + 3 I_F - 4 \sum n_i + 4$$
$$= \sum n_i d_i - B + \sum n_i b_i - \tfrac{3}{2} F + \sum \tfrac{3}{2} n_i f_i - 4 \sum n_i + 4$$

$$\boxed{D = -B - \tfrac{3}{2} F + 4 + \sum n_i \delta_i}$$

where $\boxed{\delta_i \equiv \dim \mathcal{L}_i - 4}$ is the <u>index of divergence</u> of $\mathcal{L}_i$.

D tells you how divergent a Feynman diagram is. If there is any $\delta_i > 0$ in the theory, then one can increase B and F (the # external boson and fermion lines) and still have divergent diagrams ($D > 0$) just by adding enough vertices w/ $\delta_i > 0$.

This isn't a proof, but it is suggestive that terms in the Lagrangian w/ $\delta_i > 0$ make a theory nonrenormalizable, while terms w/ $\delta_i \leq 0$ are okay.

<u>Theorem (Hepp)</u>: If there are a finite # of classes of 1PI diagrams w/ superficial degree of divergence > 0, then adding counterterms for those diagrams eliminates all divergences in the theory.

Conclusion: $\boxed{\begin{array}{ll} \dim \mathcal{L}_i \leq 4 & \rightarrow \text{renormalizable interactions} \\ \dim \mathcal{L}_i > 4 & \rightarrow \text{nonrenormalizable} \end{array}}$