

# Phys 722

Last semester you learned how to define a relativistic quantum field theory, how to calculate scattering amplitudes in perturbation theory, and how to relate the scattering amplitudes to cross sections and decay widths.

This semester will focus on techniques for calculating loop corrections to scattering amplitudes, renormalization, functional integral quantization, and gauge theories.

A prototypical field theory for studying each of these topics is Quantum Electrodynamics. Recall the QED Feynman rules:

## External Lines

$$\begin{array}{c} \uparrow \\ r_A/p_A \end{array} = u^{r_A}(p_A) \\ \text{ingoing } e^-$$

$$\begin{array}{c} r_1/p_1 \\ \uparrow \end{array} = \bar{u}^{r_1}(p_1) \\ \text{outgoing } e^-$$

$$\begin{array}{c} \downarrow \\ r_A/p_A \end{array} = \bar{v}^{r_A}(p_A) \\ \text{ingoing } e^+$$

$$\begin{array}{c} r_1/p_1 \\ \downarrow \end{array} = v^{r_1}(p_1) \\ \text{outgoing } e^+$$

Convention! For  $e^-$ , momentum is labeled along charge flow.  
For  $e^+$ , momentum is labeled opposite charge flow.

$$\left. \begin{matrix} \} \\ \mu, r, p \end{matrix} \right\} = \epsilon_{\mu}^{(r)}(p)$$

ingoing photon

$$\left. \begin{matrix} \} \\ \mu, r, p \end{matrix} \right\} = \epsilon_{\mu}^{(r)}(p)^*$$

outgoing photon

### Propagators (internal lines)

Photons:  $\mu \rightsquigarrow \nu$   
 $k \rightarrow$

$$\frac{-i g_{\mu\nu}}{k^2 + i\epsilon}$$

(will be justified later)

Fermions:

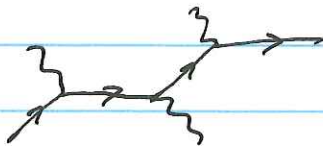
$\longrightarrow$   
 $k \rightarrow$

$$\frac{i(\not{k} + m)}{k^2 - m^2 + i\epsilon}$$

### Vertex

$\begin{matrix} \nearrow \mu \\ \nwarrow \nu \end{matrix}$   $-ie\gamma^{\mu}$

These rules are enough to calculate any tree-level diagram in QED, i.e. any diagram with no closed loops. We will deal with loops soon.



Tree-level



One-loop

## The Coulomb Potential

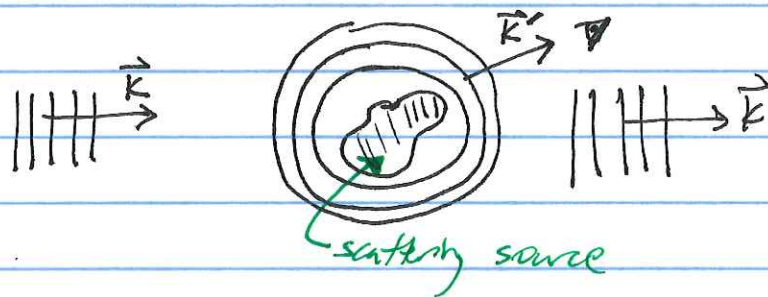
In the center-of-mass frame, two-body scattering can be interpreted as scattering off a potential. If we take the non-relativistic limit of a scattering cross section, we learn about the potential by comparison with the Born approximation in non-relativistic QM:

The solution to the Schrödinger equation w/ potential  $V(\vec{r})$  that looks like a plane wave when  $V(\vec{r}) \rightarrow 0$  is approximately,

$$\psi(\vec{r}) \approx \frac{1}{(2\pi)^{3/2}} \left[ e^{i\vec{k} \cdot \vec{r}} + \frac{e^{ikr}}{r} f(\vec{k}, \vec{k}') \right], \quad k \equiv |\vec{k}| = |\vec{k}'|$$

$\vec{k}' = k \hat{r}$

↑ plane wave                      ↑ outgoing spherical wave



In the lowest-order Born approximation,

$$f(\vec{k}, \vec{k}') \approx -\frac{m}{2\pi} \int d^3r' e^{i(\vec{k} - \vec{k}') \cdot \vec{r}'} V(\vec{r}')$$
$$\equiv -\frac{m}{2\pi} \tilde{V}(\vec{k} - \vec{k}')$$

In terms of  $f(\vec{k}, \vec{k}')$  the differential cross section is given by,

$$\frac{d\sigma}{d\Omega} = |f(\vec{k}, \vec{k}')|^2 = \frac{m^2}{4\pi^2} |\tilde{V}(\vec{k} - \vec{k}')|^2$$

If  $V(\vec{r})$  describes the potential between two particles that are scattering off of one another, then  $m$  should be replaced by the reduced mass,

$$m_{\text{red}} = \frac{m_1 m_2}{m_1 + m_2}$$

$$= \frac{m}{2} \text{ if } m_1 = m_2 = m.$$

The cross section in NRQM should be compared with our field theoretic expression,

$$\frac{d\sigma}{d\Omega} = \frac{1}{64\pi^2 \omega_{tot}^2} \frac{|\vec{P}_i|}{|\vec{P}_A|} |M|^2$$

$$\approx \frac{1}{64\pi^2 (2m)^2} |M|^2$$

in the non-rel. limit of elastic scattering of particles with equal mass  $m$ .

Comparing the two expressions, we identify

$$|\tilde{V}(\vec{k}-\vec{k}')|^2 \approx \frac{1}{16m^4} |M(\vec{k} \rightarrow \vec{k}')|^2$$

We could have done better by comparing non-rel. scattering amplitudes directly, which would have given

$$\tilde{V}(\vec{k}-\vec{k}') \approx -\frac{1}{4m^2} M(\vec{k} \rightarrow \vec{k}')$$

This expression is more useful because it carries direct information about the sign of the potential.

For electron-electron scattering to lowest order  $\mathcal{O}$

$$iM = \begin{array}{c} k_1' \\ \swarrow \\ \text{---} \\ \searrow \\ k_2' \\ \swarrow \\ k_1 \\ \swarrow \\ k_2 \end{array} + \begin{array}{c} k_1' \\ \swarrow \\ \text{---} \\ \searrow \\ k_2' \\ \swarrow \\ k_1 \\ \swarrow \\ k_2 \end{array}$$

$$= (-ie)^2 \bar{u}(k_1') \gamma^\mu u(k_1) \frac{-ig_{\mu\nu}}{(k_1 - k_1')^2} \bar{u}(k_2') \gamma^\nu u(k_2) - (k_1' \leftrightarrow k_2')$$

Consider the first term in  $iM$ .

$$\text{In the nonrel limit } \bar{u}(k_1') \gamma^0 u(k_1) = u^\dagger(k_1') u(k_1) \approx 2m \xi_1'^\dagger \xi_1,$$

where  $\xi_1$  and  $\xi_1'$  are the 2-component Pauli spinors describing the initial and final state electron labeled 1.

$$\text{Also, } \bar{u}(k_1') \gamma^i u(k_1) \rightarrow 0 \text{ if } \vec{k}_1, \vec{k}_1' \rightarrow 0.$$

The nonrelativistic scattering amplitude is, (ignoring the crossed term)

$$\begin{aligned} iM &= \frac{ie^2}{-|\vec{k}' - \vec{k}|^2} (2m \xi_1'^\dagger \xi_1) (2m \xi_2'^\dagger \xi_2) g_{00} \\ &= \frac{-ie^2}{|\vec{k}' - \vec{k}|^2} \cdot 4m^2 (\xi_1'^\dagger \xi_1) (\xi_2'^\dagger \xi_2) \end{aligned}$$

The spin of each electron is conserved, since

$$\xi_1'^\dagger \xi_1 \equiv \xi_1'^{\dagger r_1} \xi_1^{r_1} = \delta^{r_1' r_1}$$


$$\xi_2'^\dagger \xi_2 \equiv \xi_2'^{\dagger r_2} \xi_2^{r_2} = \delta^{r_2' r_2}$$

Factoring out the spin-conserving Kronecker  $\delta$ 's and comparing w/ the Born approx, we get the Fourier transformed potential,

$$\tilde{V}(\vec{k} - \vec{k}') = \frac{e^2}{|\vec{k}' - \vec{k}|^2}$$

Fourier transforming, we get the potential in coordinate space. We will consider a more general potential,  $V(\vec{k}) = \frac{e^2}{|\vec{k}|^2 + m^2}$ , and take  $m^2 \rightarrow 0$  at the end.

Then,  $V(\vec{r}) = e^2 \int \frac{d^3k}{(2\pi)^3} \frac{e^{-i\vec{k}\cdot\vec{r}}}{k^2 + m^2}$

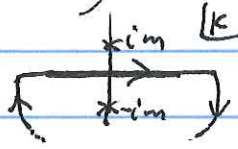


$$= e^2 \int_0^\infty dk \int_0^\pi d\theta \int_0^{2\pi} d\phi \frac{k^2}{(2\pi)^3} \sin\theta \frac{e^{-ik|\vec{r}|\cos\theta}}{k^2 + m^2}$$

$$= \frac{e^2}{(2\pi)^2} \int_0^\infty dk \frac{k^2}{k^2 + m^2} \frac{e^{-ik|\vec{r}|} - e^{ik|\vec{r}|}}{-ik|\vec{r}|}$$

$$= \frac{ie^2}{4\pi^2|\vec{r}|} \int_{-\infty}^{\infty} dk \frac{ke^{-ik|\vec{r}|}}{k^2 + m^2}$$

Analytically continuing to the complex  $k$ -plane and closing the contour in the lower-half plane



$$V(\vec{r}) = \frac{ie^2}{4\pi^2|\vec{r}|} (-2\pi i) \underset{\substack{\uparrow \\ \text{clockwise} \\ \text{contour}}}{(-im)} \frac{e^{-i(-im)|\vec{r}|}}{-2im}$$

$\uparrow$  residue at  $k = -im$

$$V(\vec{r}) = \frac{e^2}{4\pi|\vec{r}|} e^{-m|\vec{r}|}$$

Yukawa Potential

As  $m \rightarrow 0$  we recover the Coulomb potential,

$$V(\vec{r}) = \frac{e^2}{4\pi|\vec{r}|}$$

The sign of the potential teaches us that particles w/ the same charge repel in QED.

What about particle-antiparticle scattering?

$$iM \supset \begin{array}{c} k_1' \\ \swarrow \\ \text{---} \\ \searrow \\ k_2' \\ \downarrow \\ k_2 \\ \swarrow \\ k_1 \end{array} = -(ie)^2 \bar{u}(k_1') \gamma^\mu u(k_1) \frac{-ig_{\mu\nu}}{(k_1 - k_2)^2} \bar{v}(k_2) \gamma^\nu v(k_2')$$

$$\bar{v}(k_2) \gamma^0 v(k_2') = v^\dagger(k_2) v(k_2') \approx 2m \xi_2^\dagger \xi_2',$$

just as for particles w/  $v \rightarrow u$ .

The extra -ve sign came from exchanges of fermion fields:

$$\begin{array}{c} \langle k_1', k_2' | : \bar{\psi} \psi \bar{\psi} \psi : | k_1, k_2 \rangle \\ \propto \langle 0 | b_{k_2'} a_{k_1'} \bar{\psi} \psi \bar{\psi} \psi a_{k_1}^\dagger b_{k_2}^\dagger | 0 \rangle \end{array}$$

— Requires odd # exchanges of fermion ops to move the  $\psi$ 's and  $\bar{\psi}$ 's next to the corresponding creation or annihilation operators  $\rightarrow$  gives an overall (-).

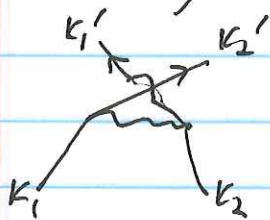
Hence, the potential between particles and antiparticles takes the same form as the potential between particles-particles (or antiparticles-antiparticles), but w/ the opposite sign.

$\rightarrow$  particles and antiparticles attract in QED



## Exchange Potentials

In the derivation of the Coulomb potential we quietly ignored one of the contributions to the scattering amplitude,



, which is equal to our previous term w/  $k_1' \leftrightarrow k_2'$ , up to a sign.

Hence, we get another term in the potential

$$\tilde{V}_{\text{exch.}}(\vec{k}_1 - \vec{k}_2') = \tilde{V}_{\text{exch.}}(\vec{k}_1 + \vec{k}_1')$$

←  $\vec{k}_2' = -\vec{k}_1'$  in CM frame.

$$= \frac{-e^2}{|\vec{k}_1 + \vec{k}_1'|^2}$$

We would not have gotten this term had we considered instead scattering of non-identical particles.

This is the exchange potential in the Hamiltonian of non-relativistic QM w/ two identical particles,

$$H = H_0 + \underbrace{V}_{\substack{\uparrow \text{Coulomb} \\ \text{Potential}}} + \underbrace{VE}_{\substack{\uparrow \text{Exchange} \\ \text{Coulomb potential}}}$$

$$E|\vec{r}_1, \vec{r}_2\rangle = -|\vec{r}_2, \vec{r}_1\rangle$$

↑  
Exchange operator

↑  
Because these are fermions.