Physics 722, Spring 2016
Problem Set 1, due Thursday, Feb 4.

1. *Derivative Interactions*

In class we mentioned the subtleties in dealing with derivative interactions. Here you will study a simple example in which the naive handling of derivatives gives the correct answer.

Consider a free real scalar field with Lagrangian,

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi)^2 - \frac{m^2}{2}\phi^2.$$

We can equivalently write this as a Lagrangian in terms of a rescaled field $\tilde{\phi} \equiv Z^{-1/2}\phi$ as,

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \tilde{\phi})^2 - \frac{m^2}{2}\tilde{\phi}^2 + (Z-1)\left[\frac{1}{2}(\partial_\mu \tilde{\phi})^2 - \frac{m^2}{2}\tilde{\phi}^2\right].$$

The Fourier-transformed two-point function for the field ϕ is the usual free-field propagator:

$$\int d^4x\, e^{ik\cdot x}\langle 0|T\left(\phi(x)\phi(0)\right)|0\rangle = \frac{i}{k^2 - m^2 + i\epsilon}.$$

a) Given the relationship between $\tilde{\phi}$ and ϕ, the propagator for the field $\tilde{\phi}$ is a simple rescaling of the propagator for the field ϕ. What is

$$\int d^4x\, e^{ik\cdot x}\langle 0|T\left(\tilde{\phi}(x)\tilde{\phi}(0)\right)|0\rangle?$$

In the remainder of this problem you will reproduce the result of part (a) by evaluating $\langle 0|T\left(\tilde{\phi}(x)\tilde{\phi}(0)\right)|0\rangle$ in perturbation theory, where you are to think of the the terms in the Lagrangian proportional to $(Z-1)$ as being interaction terms.

b) Derive the Feynman rules for the two-point vertices corresponding to the interactions in this theory, treating the derivatives in the interactions naively.

c) By summing over *all* diagrams that contribute to the Fourier transformed two-point function, $\int d^4x\, e^{ik\cdot x}\langle 0|T\left(\tilde{\phi}(x)\tilde{\phi}(0)\right)|0\rangle$, show that you recover the result of part (a).

2. *Renormalization of Spinor Fields*

In class we argued that in any Lorentz-invariant theory with a parity symmetry, the vacuum-to-one-particle matrix element of a Dirac spinor field operator $\Psi(x)$ is the same as in the free theory up to an overall rescaling that depends on the interactions.

We demonstrated this explicitly for the case that the one-particle state is spin-up in the rest frame of the particle. Repeat the argument in the case that the one-particle state is spin-down, and explain why for a generic one-particle state the vacuum-to-one-particle matrix element is the same as in the free theory up to an overall rescaling.

3. Show that for a Lorentz-scalar field $\phi(x)$,

$$\langle 0|\phi(0)|\mathbf{k}\rangle = \langle 0|\phi(0)|\mathbf{0}\rangle,$$

where $|\mathbf{0}\rangle$ is a one-particle state with vanishing spatial momentum and $|0\rangle$ is the vacuum. Recall that we used this relation in the derivation that the physical mass defined by the mass-shell condition $\omega_{\mathbf{k}}^2 - \mathbf{k}^2 = m^2$ in the interacting theory is the same as the location of the pole in the renormalized scalar-field two-point function.

Hint: Think about the unitary operation that converts $\phi(x)$ to $\phi(\Lambda^{-1}x)$ for Lorentz-transformation Λ.