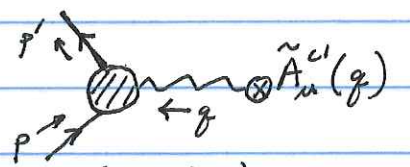
Electron Vertex Function

Consider scattering of an electron off of a background electromagnetic field. The corresponding Feynman diagrams have the form:



$(q = p' - p)$

$$\equiv -i\bar{u}(p')\,\tilde{\Gamma}^\mu(p,p')\,u(p)\,\tilde{A}^{cl}_\mu(p'-p)$$

$$\tilde{A}^{cl}_\mu(q) = \int d^4x\, e^{iq\cdot x} A^{cl}_\mu(x)$$

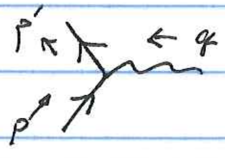
You can think of the background electromagnetic field as due to an external current j_μ.

In Lorenz gauge, $\Box A^{cl}_\mu = e j_\mu$, i.e.

$$\boxed{\tilde{A}^{cl}_\mu(q) = -\frac{e}{q^2}\,\tilde{j}_\mu(q)}$$

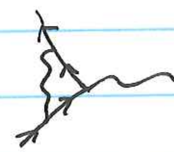
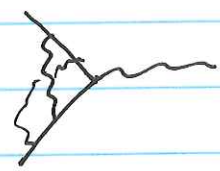
The 4x4 matrix function of p, p' $\Gamma^\mu(p,p')$ is called the electron vertex function.

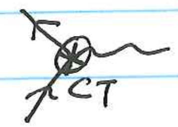
At tree level,



$$\boxed{-i\,\Gamma^{(1)\mu}(p,p') = -ie\gamma^\mu}$$

Higher order corrections to the vertex function include

$-i\tilde{\Gamma}^\mu(p,p') =$  $+$  $+$  $+ \dots$

$+$ 

The counterterm comes from $\mathcal{L}_{CT} \supset -D\bar{\Psi}e\not{A}\Psi$,

$$\boxed{-i\tilde{\Gamma}^\mu(p,p') \supset -iDe\gamma^\mu}$$

Higher order corrections will not all be proportional to γ^μ. Lorentz invariance and gauge invariance restrict the possible matrix structures appearing in $\tilde{\Gamma}^\mu$. We can decompose $\tilde{\Gamma}^\mu$ as:

$$\tilde{\Gamma}^\mu(p,p') = A\gamma^\mu + B(p'^\mu + p^\mu) + C(p'^\mu - p^\mu)$$

When $\tilde{\Gamma}^\mu(p,p')$ appears in an S-matrix element it is sandwiched between spinor wavefunctions $\bar{u}(p')$ and $u(p)$. The on-shell momenta and wavefunctions satisfy $p^2 = p'^2 = m^2$, $\not{p}u(p) = mu(p)$

$\bar{u}(p')\not{p}' = m\bar{u}(p')$

We can think of A, B, C as functions of $(p'-p)^2$, as any other Lorentz invariants are redundant, ie

$A = A(q^2)$, $B = B(q^2)$, $C = C(q^2)$, $q^\mu = p'^\mu - p^\mu$.

(Note that we didn't include matrix structures including γ_5 because they would violate parity.)

Another Ward Identity due to gauge invariance implies that $\boxed{\text{inside S-matrix elements, } q_\mu \tilde{\Gamma}^\mu = 0}$

$$0 = q_\mu \, \bar{u}(p') \tilde{\Gamma}^\mu(p,p') u(p)$$

$$= A(q^2) \, \bar{u}(p') (\not{p}' - \not{p}) u(p) + B(q^2) \, \bar{u}(p') u(p) (p'^\mu - p^\mu)(p'_\mu + p_\mu) + C(q^2) \, q^2$$

Since $\bar{u}(p') \not{p} u(p) = \bar{u}(p') \not{p}' u(p)$, the term $\propto A(q^2)$ vanishes. Since $p^2 = p'^2 = m^2$, the term proportional to $B(q^2)$ vanishes.

If the photon is not on-shell, $q^2 \neq 0$. Hence, the Ward Identity implies $C(q^2) = 0$. Hence $\tilde{\Gamma}^\mu$ takes the form,

$$\boxed{\tilde{\Gamma}^\mu(p,p') = A(q^2) \gamma^\mu + B(q^2)(p'^\mu + p^\mu)}$$

It is customary to rewrite the second term using an identity for solutions to the Dirac equation $u(p)$, $u(p')$:

$$\boxed{\bar{u}(p') \gamma^\mu u(p) = \frac{1}{2m} \bar{u}(p') \left(p'^\mu + p^\mu + i \sigma^{\mu\nu} q_\nu \right) u(p)}$$

<u>Gordon identity</u>

where $\sigma^{\mu\nu} = \frac{i}{2} [\gamma^\mu, \gamma^\nu]$

The Gordon identity follows from

$$\gamma^\mu \not{p} = \tfrac{1}{2}\{\gamma^\mu, \not{p}\} + \tfrac{1}{2}[\gamma^\mu, \not{p}]$$
$$= p^\mu - i\sigma^{\mu\nu} p_\nu$$

and $\not{p}' \gamma^\mu = p'^\mu + i\sigma^{\mu\nu} p'_\nu$. Then,

$$\bar{u}(p')\gamma^\mu u(p) = \frac{1}{2m}\bar{u}'\left(\not{p}'\gamma^\mu + \gamma^\mu \not{p}\right)u$$

$$= \frac{1}{2m}\bar{u}(p')\left(p'^\mu + p^\mu + i\sigma^{\mu\nu} q_\nu\right)u(p) \quad \text{as claimed.}$$

The part $\propto p'^\mu + p^\mu$ has the form of the term in $\bar{u}\tilde{\Gamma}^\mu u$ multiplying $B(q^2)$. Hence, instead of using γ^μ and $(p'^\mu + p^\mu)$ in the decomposition of $\tilde{\Gamma}^\mu$, we can use γ^μ and $i\sigma^{\mu\nu}q_\nu$:

$$\boxed{\tilde{\Gamma}^\mu(p,p') = e\gamma^\mu F_1(q^2) + \frac{i\sigma^{\mu\nu}q_\nu}{2m}\, e F_2(q^2)}$$

Our goal is to interpret the <u>form factors</u> $F_1(q^2)$ and $F_2(q^2)$ and calculate them at 1-loop.

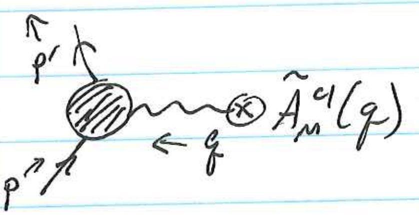
$F_1(q^2) \equiv$ Dirac Form Factor
$F_2(q^2) \equiv$ Pauli Form Factor.

At tree level, $F_1(q^2) = 1$, $F_2(q^2) = 0$.

To understand the physical meaning of the form factors we consider scattering of a <u>nonrelativistic</u> electron from a background electromagnetic field.

First, consider a background electrostatic potential $\phi(\vec{x})$, s.t. $A_\mu^{cl}(x) = (\phi(\vec{x}), \vec{0})$.

$$\tilde{A}_\mu^{cl}(q) = \int d^4x\, e^{iq\cdot x} A_\mu^{cl}(x) = \left(2\pi\delta(q^0)\tilde{\phi}(\vec{q}), \vec{0}\right)$$



$$iM = -i\,\bar{u}(p')\,\tilde{\Gamma}^\mu(p,p')u(p)\,\tilde{A}_\mu^{cl}(p'-p)$$
$$= -i\,\bar{u}(p')\,\tilde{\Gamma}^0(p,p')\,u(p)\,\tilde{\phi}(\vec{p}\,'-\vec{p})$$

(with the $2\pi\delta(q^0)$ factored out)

If the electrostatic field varies slowly over a large region, then $\tilde{\phi}(\vec{q})$ will be concentrated at $\vec{q} \simeq \vec{0}$.

In that limit, $\tilde{\Gamma}^0(p,p') = e\gamma^0 F_1(q^2) + \frac{i\sigma^{0j}}{2m} e q_j F_2(q^2)$

$$\rightarrow e\gamma^0 F_1(q^2)$$

In the nonrelativistic limit $\bar{u}(p')\gamma^0 u(p) = 2m\,\xi'^\dagger\xi$. The scattering amplitude becomes,

$$\boxed{iM \approx -ie F_1(0)\,\tilde{\phi}(\vec{q})\cdot 2m\,\xi'^\dagger\xi}$$

This agrees with the Born approximation for scattering off of a potential $\boxed{V(\vec{x}) = eF_1(0)\,\phi(\vec{x}).}$

$\boxed{\text{Hence, } eF_1(0) \text{ is the electric charge of the electron.}}$
Since $F_1(0) = 1$ at tree level, radiative corrections to $F_1(q^2)$ should vanish at $q^2 = 0$. → Renormalization Condition

Now consider scattering off of a magnetic field specified by a static vector potential $A_\mu^{cl}(x) = (0, \vec{A}^{cl}(\vec{x}))$.

Then the scattering amplitude is,

$$iM = \sum_j ie\left[\bar{u}(p')\left(\gamma^j F_1(q^2) + \frac{i\sigma^{j\nu}q_\nu}{2m}F_2(q^2)\right)u(p)\right]\tilde{A}^j_{cl}(\vec{q})$$

Again we assume that $\tilde{A}^j_{cl}(\vec{q})$ is peaked near $\vec{q} = \vec{0}$.

Consider the spinors in the Weyl basis, expanded in $\vec{p}$ and $\vec{p}'$:

$$u(p) = \begin{pmatrix} \sqrt{p\cdot\sigma}\,\xi \\ \sqrt{p\cdot\bar{\sigma}}\,\xi \end{pmatrix} \approx \sqrt{m}\begin{pmatrix} \left(1 - \frac{\vec{p}\cdot\vec{\sigma}}{2m}\right)\xi \\ \left(1 + \frac{\vec{p}\cdot\vec{\sigma}}{2m}\right)\xi \end{pmatrix}$$

Consider the $F_1(q^2)$ term in iM:

$$iM \supset ie\,\bar{u}(p')\gamma^j F_1(q^2)u(p)\,\tilde{A}^j_{cl}(\vec{q})$$

Exercise: $$\approx ie\cdot 2m\,\xi'^{\dagger}\left(\frac{\vec{p}'\cdot\vec{\sigma}}{2m}\sigma^j + \sigma^j\frac{\vec{p}\cdot\vec{\sigma}}{2m}\right)\xi\,F_1(0)\tilde{A}^j_{cl}(\vec{q})$$

Use $\sigma^i \sigma^j = \delta^{ij} + i \epsilon^{ijk} \sigma^k$ (summed over repeated indices)

$$i\mathcal{M} \approx ie \frac{2m}{2m} \xi'^{\dagger} (\vec{p}\,' + \vec{p}) \cdot \vec{\tilde{A}}_{cl}(\vec{q}) \xi F_1(0)$$

$$+ ie \frac{2m}{2m} \xi'^{\dagger} (+i \epsilon^{ijk} q^j \sigma^k) \xi \tilde{A}^i_{cl}(\vec{q}) F_1(0)$$

The first term, w/ $\vec{p}\,' \approx \vec{p}$, corresponds to the $(\vec{p} \cdot \vec{A} + \vec{A} \cdot \vec{p})$ term in the nonrelativistic Hamiltonian.

The second term contributes to the electron's magnetic moment interaction:

The magnetic field is $\vec{B} = \nabla \times \vec{A}_{cl}$. Fourier transform:

$$\tilde{B}^k(\vec{q}) = \int d^3x \, e^{+i\vec{q}\cdot\vec{x}} \epsilon_{kij} \partial_i A^{cl}_j(\vec{x})$$
$$= -iq^i \epsilon^{kij} \int d^3x \, e^{+i\vec{q}\cdot\vec{x}} A^j_{cl}(\vec{x})$$

(Euclidean conventions — repeated indices summed, no minus signs)

(integration by parts)

$$= -i \epsilon^{kij} q^i \tilde{A}^j_{cl}(\vec{q})$$

Then the second term in $i\mathcal{M}$ above is:

$$i\mathcal{M} \approx -ie \, \xi'^{\dagger} \sigma^k \xi \, \tilde{B}^k(\vec{q}) \, F_1(0)$$

This is the Born approximation to scattering off a potential,

$$V(\vec{x}) = -\vec{\mu}_1 \cdot \vec{B} \quad , \quad \boxed{\vec{\mu}_1 = \frac{e}{m} F_1(0) \, \xi'^{\dagger} \frac{\vec{\sigma}}{2} \xi}$$

The $F_2(q^2)$ term in $i\mathcal{M}$ contributes similarly:

$$i\mathcal{M} \supset +\frac{e}{2m} \cdot \bar{u}(p') \frac{i}{2}[\gamma^j, \gamma^k] q^k F_2(q^2) u(p) \tilde{A}^j_{cl}(\vec{q})$$

Use again $\sigma^i \sigma^j = \delta^{ij} + i\epsilon^{ijk}\sigma^k$

$$\rightarrow [\sigma^i, \sigma^j] = 2i\epsilon^{ijk}\sigma^k$$

Then $\bar{u}(p') \sigma^{jk} q^k u(p) \approx 2m\, \xi'^{\dagger} \epsilon^{jki} q^k \sigma^i \xi$

$$i\mathcal{M} \approx \frac{e}{\cancel{2m}} \cdot \cancel{2m}\, \xi'^{\dagger} \sigma^i \xi\, \epsilon^{jki} q^k F_2(q^2) \tilde{A}^j_{cl}(\vec{q})$$

$$= -ie\, \xi'^{\dagger} \sigma^i \xi\, \tilde{B}^i(\vec{q})\, F_2(q^2)$$

Comparing again w/ the Born approximation, we get another contribution to the magnetic moment,

$$\vec{\mu}_2 = \frac{e}{m} F_2(0)\, \xi'^{\dagger} \frac{\vec{\sigma}}{2} \xi$$

where we set $q^2 = 0$ in $F_2(q^2)$ because we assume $\tilde{A}_{cl}(\vec{q})$ is dominated by $\vec{q} \approx \vec{0}$.

Adding everything together, we find a magnetic moment interaction,

$$\boxed{\begin{aligned} V(\vec{x}) &= -\vec{\mu} \cdot \vec{B}(\vec{x}) \\ \vec{\mu} &= \frac{e}{m}\left[F_1(0) + F_2(0)\right] \xi'^{\dagger} \frac{\vec{\sigma}}{2} \xi \end{aligned}}$$

$\xi'^{\dagger}\frac{\vec{\sigma}}{2}\xi$ is the electron spin $\vec{S}$. It is common to write the electron magnetic moment in the form,

$$\vec{\mu} = g\left(\frac{e}{2m}\right)\vec{S}\,, \quad g \equiv \text{Landé g-factor.}$$

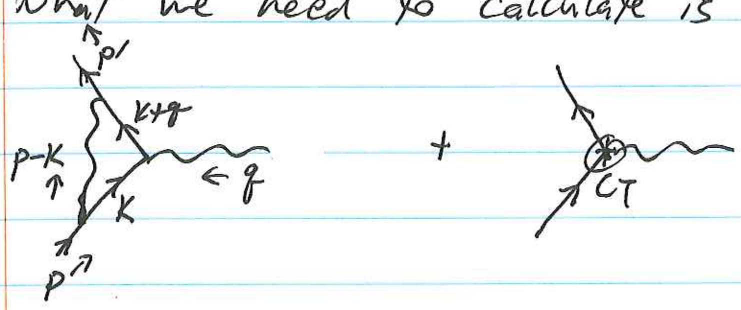
Since we have argued that $F_1(0) = 1$, we have derived,

$$\boxed{g = 2 + 2F_2(0)}$$

$g = 2$ was the prediction of the Dirac equation. At tree level $F_2(0) = 0$, so $F_2(0) = \mathcal{O}(\alpha)$.

The value of $(g-2)$ is called the <u>anomalous magnetic moment of the electron</u>.

What we need to calculate is



The counterterm is chosen so that the electron charge is e, so we expect the same counterterm which set $\frac{d\Sigma}{d\not{p}}\big|_{\not{p}=m} = 0$ for the electron self energy to also set $F_1(0) = 1$. This is a consequence of a Ward Identity, but we will have to wait to see this.