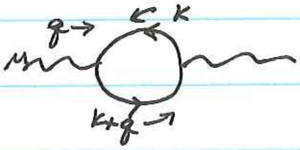
Photon Self-Energy: The Calculation

We need to calculate



$\equiv i\Pi^{\mu\nu}(q)$.

$$i\Pi^{\mu\nu}(q) = (-ie)^2 \overset{\substack{\text{from fermion}\\ \text{loop}}}{\swarrow}(-\mathrm{Tr})\left[\int\frac{d^4k}{(2\pi)^4}\,\gamma^\mu\,\frac{i(\not k+m)}{k^2-m^2+i\epsilon}\,\gamma^\nu\,\frac{i(\not k+\not q+m)}{(k+q)^2-m^2+i\epsilon}\right]$$

Use
$$\mathrm{Tr}\,\gamma^\mu\gamma^\nu = 4g^{\mu\nu}$$
$$\mathrm{Tr}\,\gamma^\mu\gamma^\nu\gamma^\lambda = 0$$
$$\mathrm{Tr}\,\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma = 4(g^{\mu\nu}g^{\rho\sigma} - g^{\mu\rho}g^{\nu\sigma} + g^{\mu\sigma}g^{\nu\rho})$$

$$i\Pi^{\mu\nu}(q) = -e^2\int\frac{d^4k}{(2\pi)^4}\,4\left[\frac{k^\mu(k+q)^\nu - g^{\mu\nu}k\cdot(k+q) + k^\nu(k+q)^\mu + g^{\mu\nu}m^2}{(k^2-m^2+i\epsilon)((k+q)^2-m^2+i\epsilon)}\right]$$

Combine denominators a la Feynman:

$$\left[(k^2-m^2+i\epsilon)((k+q)^2-m^2+i\epsilon)\right]^{-1} = \int_0^1 dx\left[k^2+2x\,k\cdot q + xq^2 - m^2 + i\epsilon\right]^{-2}$$

Complete the square: $\ell^\mu = k^\mu + xq^\mu$

$$\longrightarrow\ = \int_0^1 dx\left[\ell^2 + x(1-x)q^2 - m^2 + i\epsilon\right]^{-2}$$

Then,

$$i\Pi^{\mu\nu}(q) = -e^2\cdot 4\int_0^1 dx\int\frac{d^4\ell}{(2\pi)^4}\Big[2\ell^\mu\ell^\nu - g^{\mu\nu}\ell^2 + 2x^2q^\mu q^\nu - 2xq^\mu q^\nu + g^{\mu\nu}(m^2+xq^2-x^2q^2)\Big]\left[\ell^2+x(1-x)q^2-m^2+i\epsilon\right]^{-2}$$

(We have thrown away terms linear in ℓ^μ which integrate to zero.)

Regularization

In order to make divergent integrals that appeared at intermediate stages of our self energy calculations finite, we introduced a hard momentum cutoff $k_E^2 = \Lambda^2$, and took $\Lambda^2 \to \infty$ at the end of the calculation. The hard momentum cutoff violates Lorentz invariance and gauge invariance, and will therefore sometimes lead us astray, even producing nonsensical results.

There are other regularization methods better suited for some theories. As long as the regularization procedure respects the symmetries of the theory, most results will be independent of the regularization method. When they are not, the form of the cutoff must be included as an axiom of the field theory.

Two useful regularization methods are known as regulator fields and dimensional regularization. Here we discuss dimensional regularization, which we will apply to our calculation of the photon self energy.

Dimensional Regularization

Loop integrals are finite in a small enough number of dimensions. Dim Reg is the procedure of analytically continuing integrals in the number of dimensions, and picking out divergences as $d \to 4$.

Consider integrals of the form

$$I = \int \frac{d^d p_E}{(p_E^2 + q^2)^n}$$, as might appear after Wick rotating a loop integral.

I is convergent if $n > \frac{d}{2}$, $d = \#$ Euclidean dimensions.

The trick is to turn the denominator into an exponential using the Gamma function,

$$\Gamma(n) = \int_0^\infty dt \; t^{n-1} e^{-t}$$

Change variable $t = \alpha\lambda$, α real > 0.

$$\Gamma(n) = \int_0^\infty d(\alpha\lambda) \; (\alpha\lambda)^{n-1} e^{-\alpha\lambda}$$, i.e.

$$\frac{1}{\alpha^n} = \frac{1}{\Gamma(n)} \int_0^\infty d\lambda \; \lambda^{n-1} e^{-\alpha\lambda}$$

Then we can represent our integral as,

$$I = \frac{1}{\Gamma(n)} \int_0^\infty d\lambda\, \lambda^{n-1} \underbrace{\int d^dK\, e^{-\lambda(K^2+a^2)}}_{e^{-\lambda a^2}\left(\frac{\pi}{\lambda}\right)^{d/2}}$$

$$= \frac{\pi^{d/2}}{\Gamma(n)} \int_0^\infty \lambda^{n-d/2-1}\, e^{-\lambda a^2}\, d\lambda$$

Finally, $\boxed{I = \int \frac{d^dP_E}{(P_E^2+a^2)^n} = \frac{\pi^{d/2}}{\Gamma(n)} \frac{\Gamma(n-d/2)}{a^{2n-d}}}$

'tHooft and Veltman's trick was to analytically continue this formula to arbitrary complex d.

Away from even integers $d \geq 2n$, the Gamma function is well defined.

As $d \to 4$ poles in $(d-4)$ appear, which cancel in convergent combinations.

To <u>renormalize</u> using dim reg, calculate in arbitrary d, include counterterms and impose renormalization conditions, and set $d \to 4$ <u>at the very end.</u> You have to be careful w/ continuing the theory to arbitrary dimensions: the dimensions of couplings depend on d, ie $\frac{e^2}{4\pi} \neq \frac{1}{137}$ in $d \neq 4$.

To pick out the poles as $d \to 4$ use the expansion of the Gamma function about negative integers (or zero):

For $n \geq 0$, $\boxed{\Gamma(-n+\epsilon) = \frac{(-1)^n}{n!}\left[\frac{1}{\epsilon} + \underset{\nwarrow \text{some finite \#}}{\psi(n+1)} + \mathcal{O}(\epsilon)\right]}$

(which will not appear in physical results)

$$\psi(1) = -\underset{\uparrow \text{Euler-Mascheroni constant}}{\gamma_E} = -0.5772\ldots$$

Note: It is important to be careful with powers that go to zero as $d \to 4$.

$$A^{4-d} = \exp[(4-d)\log A]$$

$$\approx 1 + \underline{\underline{(4-d)\log A}} + \mathcal{O}(4-d)^2$$

If A^{4-d} multiplies a pole from a Γ-function $\propto \frac{1}{4-d}$, then you would miss the finite term $\propto \log A$ if you set $d=4$ too quickly.

Dimensional Regularization with γ-matrices

For Lorentz invariance we require the Clifford algebra

$$\boxed{\begin{aligned}\{\gamma^\mu,\gamma^\nu\} &= 2g^{\mu\nu}\\ \mathrm{Tr}\,\underline{1} &= 4\end{aligned}} \leftarrow \text{or } 2^{d/2} \text{ in even dims, } 2^{\frac{d-1}{2}} \text{ in odd dims}$$

The metric satisfies $\boxed{g_{\mu\nu}g^{\mu\nu} = d}$

From the Clifford algebra we deduce the γ-matrix contraction identities in d dimensions:

$$\gamma^\mu\gamma_\mu = d$$

$$\gamma^\mu\gamma^\nu\gamma_\mu = -(d-2)\gamma^\nu$$

$$\gamma^\mu\gamma^\nu\gamma^\rho\gamma_\mu = 4g^{\nu\rho} - (4-d)\gamma^\nu\gamma^\rho$$

$$\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma\gamma_\mu = -2\gamma^\sigma\gamma^\rho\gamma^\nu + (4-d)\gamma^\nu\gamma^\rho\gamma^\sigma$$

Again, it is important not to throw out the terms that vanish as $d\to 4$ until the end of the calculation, as they may multiply poles and become nonzero and finite.

There are different ways to handle γ^5. 't Hooft and Veltman set $\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3$, just like in $d=4$. Then $\{\gamma^5,\gamma^\mu\}=0$ for $\mu=0,1,2,3$, but not for $\mu>3$.

Alternatively, we can treat γ^5 formally as satisfying $\{\gamma^5,\gamma^\mu\}=0\ \forall\mu$.

Now back to the photon self energy.
We have derived the following expression at one-loop:

$$\equiv i\Pi^{\mu\nu}(q)$$

$$= -e^2 \cdot 4 \int_0^1 dx \int \frac{d^4 l}{(2\pi)^4} \Big[2l^\mu l^\nu - g^{\mu\nu} l^2 + 2x^2 q^\mu q^\nu - 2x q^\mu q^\nu$$

$$+ g^{\mu\nu}(m^2 + xq^2 - x^2 q^2)\Big] \left[l^2 + x(1-x)q^2 - m^2 + i\epsilon \right]^{-2}$$

In d-dimensions we can replace $l^\mu l^\nu \to \frac{1}{d} l^2 g^{\mu\nu}$.
(The $g^{\mu\nu}$ is from Lorentz invariance and symmetry in $\mu \leftrightarrow \nu$.
The $\frac{1}{d}$ is from taking the trace of both sides.)

Then, (can set $d=4$ here because it comes in a combination that doesn't vanish as $d \to 4$.)

$$i\Pi^{\mu\nu}(q) = -4e^2 \int_0^1 dx \int \frac{d^4 l}{(2\pi)^4} \Big[\left(\tfrac{2}{4} - 1\right) l^2 g^{\mu\nu} + 2x(x-1) q^\mu q^\nu$$

$$+ g^{\mu\nu}(m^2 + q^2 x(1-x))\Big] \left[l^2 + x(1-x)q^2 - m^2 + i\epsilon \right]^{-2}$$

Wick rotate: $d^4 l \to i\, d^4 l_E$, $l^2 \to -l_E^2$

$$i\pi^{\mu\nu}(q) = -4ie^2 \int_0^1 dx \int \frac{d^4 l_E}{(2\pi)^4} \frac{\left[\frac{1}{2} g^{\mu\nu} l_E^2 - 2x(1-x) q^\mu q^\nu + g^{\mu\nu}(m^2 + x(1-x)q^2)\right]}{\left[-l_E^2 + x(1-x)q^2 - m^2 + i\epsilon\right]^2}$$

We use dimensional regularization to regulate the divergent integral.

In d dimensions the electric charge e has mass dimension $[e] = \frac{4-d}{2}$, so we replace e by $e\mu^{\frac{4-d}{2}}$ for some arbitrary mass scale μ.

Check: $[\int d^dx\ \bar{\psi}\, i\not{\partial}\psi] = -d + 2[\psi] + 1 = 0$

$$\rightarrow [\psi] = \frac{d-1}{2}$$

$$[\int d^dx\ F_{\mu\nu}F^{\mu\nu}] = -d + 2[A^\mu] + 2 = 0$$

$$\rightarrow [A^\mu] = \frac{d-2}{2}$$

$$[\int d^dx\ e\,\bar{\psi}\not{A}\psi] = [e] + 2\cdot\frac{d-1}{2} + \frac{d-2}{2} - d = 0$$

$$\rightarrow [e] = \frac{4-d}{2}$$

We use our dim reg integral table:

$$\boxed{\begin{aligned}&\int\frac{d^d l_E}{(l_E^2+a^2)^n} = \frac{\pi^{d/2}}{\Gamma(n)}\frac{\Gamma(n-d/2)}{a^{2n-d}}\\ &\int\frac{d^d l_E\, l_E^2}{(l_E^2+a^2)^n} = \frac{\pi^{d/2}}{\Gamma(n)}\,\frac{d}{2}\cdot\frac{\Gamma(n-\frac{d}{2}-1)}{a^{2n-d-2}}\end{aligned}}$$

We already showed the first of these integrals. The second is left as an exercise.

It is conventional (but not necessary) to replace the $\frac{1}{(2\pi)^4}$ in the integral by $\frac{1}{(2\pi)^d}$.

We now have,

$$i\Pi^{\mu\nu}(q) = -4ie^2\mu^{4-d}\int_0^1 dx\Big[\overset{(1-\frac{2}{d})}{\frac{1}{2}}g^{\mu\nu}\cdot\frac{1}{(4\pi)^{d/2}}\,\frac{\Gamma(2-d/2-1)}{\Gamma(2)}\cdot\frac{d}{2}$$

$$(m^2-x(1-x)q^2)^{\frac{2+d-4}{2}}$$

$$+\left[g^{\mu\nu}(m^2+x(1-x)q^2)-2x(1-x)q^\mu q^\nu\right]\frac{1}{(4\pi)^{d/2}}\,\frac{\Gamma(2-d/2)}{\Gamma(2)}$$

$$\cdot(m^2-x(1-x)q^2)^{\frac{d-4}{2}}\Big]$$

$$= -4ie^2\mu^{4-d}\int_0^1 dx\Big[\frac{1}{(4\pi)^{d/2}}\cdot\Gamma(2-d/2)\,(m^2-x(1-x)q^2)\,g^{\mu\nu}$$

$$+\frac{1}{(4\pi)^{d/2}}\,\Gamma(2-d/2)\big(g^{\mu\nu}(m^2+x(1-x)q^2)-2x(1-x)q^\mu q^\nu\big)\Big]$$

$$\times\,(m^2-x(1-x)q^2)^{\frac{d-4}{2}}$$

$$= -4ie^2\int_0^1 dx\,\frac{1}{(4\pi)^{d/2}}\,\Gamma(2-d/2)\;2x(1-x)\left(\frac{\mu^2}{m^2-x(1-x)q^2}\right)^{\frac{4-d}{2}}$$

$$\times\,(g^{\mu\nu}q^2-q^\mu q^\nu)$$

Note that our result is transverse. Dim reg is a gauge invariant regulator, and preserves consequences of gauge invariance.

We define $i\Pi^{\mu\nu}(q) = i(g^{\mu\nu}q^2-q^\mu q^\nu)\,\Pi(q^2)$

$$\boxed{\Pi(q^2) = -\frac{8e^2}{(4\pi)^{d/2}}\,\Gamma(2-d/2)\int_0^1 dx\; x(1-x)\left(\frac{\mu^2}{m^2-x(1-x)q^2}\right)^{\frac{4-d}{2}}}$$

To pick out the divergence we expand $\Gamma(2-d/2)$:

$$\Gamma(2-d/2) = \frac{2}{4-d} - \gamma_E + \mathcal{O}(4-d)$$

$$\left(\frac{\mu^2}{m^2-x(1-x)q^2}\right)^{\frac{4-d}{2}} = 1 + \frac{4-d}{2}\log\left(\frac{\mu^2}{m^2-x(1-x)q^2}\right) + \mathcal{O}(4-d)^2$$

$$\Gamma(2-d/2)\left(\frac{\mu^2}{m^2-x(1-x)q^2}\right)^{\frac{4-d}{2}} = \frac{2}{4-d} - \gamma_E + \log\left(\frac{\mu^2}{m^2-x(1-x)q^2}\right) + \mathcal{O}(4-d)$$

Then,

$$\boxed{\Pi(q^2) = -\frac{8e^2}{(4\pi)^2}\int_0^1 dx\; x(1-x)\left(\frac{2}{4-d} - \gamma_E - \log\left(\frac{m^2-x(1-x)q^2}{\mu^2}\right)\right) + \mathcal{O}(4-d)}$$

The counterterm $\mathcal{L}_{CT} \supset -\frac{A}{4}F_{\mu\nu}F^{\mu\nu}$ shifts $\Pi(q^2)$ by a constant. This allows us to satisfy our renormalization condition $\tilde{\Pi}(0)=0$:

$$\boxed{\tilde{\Pi}(q^2) = \Pi(q^2) - \Pi(0) = +\frac{e^2}{2\pi^2}\int_0^1 dx\; x(1-x)\log\left(\frac{m^2-x(1-x)q^2}{m^2}\right)}$$