Fermion Masses (Cheng + Li ch. 11.2)

Standard Model fermion masses are not consistent with the SU(2)×U(1) gauge invariance of the electroweak interactions. Mass terms are of the form $m\bar{\psi}_L \psi_R + h.c.$, but the left-handed fermions are SU(2) doublets while the right-handed fermions are singlets.

After spontaneous breaking of the SU(2)×U(1) gauge invariance, Yukawa couplings give rise to fermion masses.

Recall the SU(2)×U(1) charges of the quarks and the Higgs:

$$\left.\begin{array}{ll} Q_L & 2_{1/3} \\ u_R & 1_{4/3} \\ d_R & 1_{-2/3} \\ \Phi & \underset{\text{dimension of SU(2) rep.}\nearrow}{2}\ \underset{\nwarrow\text{hypercharge}}{{}_1} \end{array}\right\} \quad \mathcal{L} \supset -\underset{\substack{\uparrow \\ \text{constant}}}{f^{(d)}}\, \bar{Q}_L \Phi d_R + h.c. \quad \text{is gauge invariant.}$$

Expanding Φ about its VEV,

$$\mathcal{L} \supset -f^{(d)} (\bar{u}_L, \bar{d}_L)\begin{pmatrix} 0 \\ v/\sqrt{2} \end{pmatrix} d_R + h.c. = -\frac{f^{(d)} v}{\sqrt{2}}\, \bar{d}_L d_R + h.c.$$

This looks like a mass term for the d quark.

To give mass to the u quark we use pseudoreality of SU(2) representations.

A group representation is called pseudoreal if $\exists$ a nonsingular matrix S such that $ST^aS^{-1}=-T^{a*}$. In that case a representation is equivalent to its conjugate.

Consider the spin-$\frac{1}{2}$ representation of $SU(2)$:

$T^1=\frac{\sigma^1}{2}$, $T^2=\frac{\sigma^2}{2}$, $T^3=\frac{\sigma^3}{2}$. Let $\boxed{S=-i\sigma^2}$.

$$ST^1S^{-1}=\sigma^2\frac{\sigma^1}{2}\sigma^2=-(\sigma^2)^2\frac{\sigma^1}{2}=-\frac{\sigma^1}{2}=-T^{1*}$$

$$ST^2S^{-1}=\sigma^2\frac{\sigma^2}{2}\sigma^2=\frac{\sigma^2}{2}=-T^{2*}$$

$$ST^3S^{-1}=\sigma^2\frac{\sigma^3}{2}\sigma^2=-\frac{\sigma^3}{2}=-T^{3*}$$

Hence, $ST^aS^{-1}=-T^{a*}$ for $a=1,2,3$.

Under an $SU(2)_w$ gauge transformation $\Phi\to e^{i\theta^a\frac{\sigma^a}{2}}\Phi$

$$\Phi^*\to e^{-i\theta^a\frac{\sigma^{a*}}{2}}\Phi^*$$

Consider $\boxed{\tilde{\Phi}=i\sigma^2\Phi^*}$

$$\hat{\Phi}\to i\sigma^2e^{-i\theta^a\frac{(\sigma^a)^*}{2}}\Phi^*$$
$$=i\sigma^2\sum_n\frac{1}{n!}\left(-i\frac{\theta^a}{2}(\sigma^a)^*\right)^n\Phi^*$$

Use $(i\sigma^2)(-\sigma^a)^*(-i\sigma^2)=\sigma^a$ and $(-i\sigma^2)(i\sigma^2)=1$ to commute the $i\sigma^2$ past the factors of $\left(-i\frac{\theta^a}{2}(\sigma^a)^*\right)$.

$$\tilde{\Phi} = i\sigma^2 \Phi^* \longrightarrow e^{i\theta^a \frac{\sigma^a}{2}} \tilde{\Phi}$$

we have shown that if Φ transforms as a doublet, then so does $\tilde{\Phi}$.

The VEV of $\tilde{\Phi}$ in our basis is $\langle \tilde{\Phi} \rangle = i\sigma^2 \begin{pmatrix} 0 \\ v/\sqrt{2} \end{pmatrix}$

$$= \begin{pmatrix} v/\sqrt{2} \\ 0 \end{pmatrix}.$$

Under $U(1)_Y$, $\tilde{\Phi}$ transforms as the conjugate to Φ, with hypercharge -1.

The Yukawa coupling $\mathcal{L} \supset -f^{(u)} \bar{Q}_L \tilde{\Phi} u_R \underset{+h.c.}{}$ is gauge invariant. After $SU(2) \times U(1)$ breaking,

$$\mathcal{L} \supset -f^{(u)} (\bar{u}_L, \bar{d}_L) \begin{pmatrix} v/\sqrt{2} \\ 0 \end{pmatrix} \underset{+h.c.}{u_R} = -\frac{f^{(u)} v}{\sqrt{2}} \bar{u}_L u_R + h.c.$$

This looks like a mass term for the u quark.

In the absence of a right-handed neutrino, the set of Yukawa couplings are (for one generation):

$$\boxed{\mathcal{L}_{Yuk} = -f^{(e)} \bar{L}_L \Phi e_R - f^{(u)} \bar{Q}_L \tilde{\Phi} u_R - f^{(d)} \bar{Q}_L \Phi d_R + h.c.}$$

If there is a right-handed neutrino, we can also include a term $-f^{(\nu)} \bar{L}_L \tilde{\Phi} \nu_R$ which gives the neutrino a mass.

3 Generations : Gauge vs. Mass Eigenstates (Cheng + Li ch. 11.3)

The Standard Model fermions are replicated in sets of 3. We label the families as follows:

$$\bar{L}_{LA} \equiv \begin{pmatrix} \nu'_{LA} \\ e'_{LA} \end{pmatrix} = \begin{pmatrix} \nu'_e, \nu'_\mu, \nu'_\tau \\ e', \mu', \tau' \end{pmatrix}_L$$

, A = family label.

☆ The prime represents gauge eigenstates here.

$$\bar{Q}_{LA} \equiv \begin{pmatrix} u'_{LA} \\ d'_{LA} \end{pmatrix} = \begin{pmatrix} u', c', t' \\ d', s', b' \end{pmatrix}_L$$

$$e'_{RA} = (e'_R, \mu'_R, \tau'_R)$$
$$u'_{RA} = (u'_R, c'_R, t'_R)$$
$$d'_{RA} = (d'_R, s'_R, b'_R)$$
$$\left(\nu'_{RA} = (\nu'_e, \nu'_\mu, \nu'_\tau)_R \right)$$

Gauge couplings:

$$\mathcal{L} \supset \bar{L}_{LA}\left(i\not{\partial} + \frac{g}{2}\vec{\sigma}\cdot\vec{A} - \frac{g'}{2}\not{B}\right)L_{LA} + \bar{e}'_{RA}\left(i\not{\partial} - g'\not{B}\right)e_{RA}$$

$$+ \bar{Q}_{LA}\left(i\not{\partial} + \frac{g}{2}\vec{\sigma}\cdot\vec{A} + \frac{g'}{6}\not{B}\right)Q_{LA} + \bar{u}'_{RA}\left(i\not{\partial} + \frac{2g'}{3}\not{B}\right)u'_{RA}$$

$$+ \bar{d}'_{RA}\left(i\not{\partial} - \frac{g'}{3}\not{B}\right)d'_{RA}$$

The gauge interactions are diagonal in terms of the gauge eigenstates.

The Yukawa couplings are not diagonal in terms of the gauge eigenstates. Assuming $\nexists\ \nu_R$:

$$-\mathcal{L}_{Yuk} = f^{(e)}_{AB}\, \bar{L}_{LA}\, \Phi\, e'_{RB} + f^{(u)}_{AB}\, \bar{Q}_{LA}\, \tilde{\Phi}\, u_{RB} + f^{(d)}_{AB}\, \bar{Q}_{LA}\, \Phi\, d_{RB} + h.c.$$

Replacing Φ by its VEV, we get the mass terms

$$-\mathcal{L}_{Yuk} \supset \frac{v}{\sqrt{2}}\left(f^{(e)}_{AB}\, \bar{e}'_{LA}\, e'_{RB} + f^{(u)}_{AB}\, \bar{u}'_{LA}\, u'_{RB} + f^{(d)}_{AB}\, \bar{d}'_{LA}\, d_{RB}\right) + h.c.$$

In the gauge eigenstate basis, the fermion mass matrices are

$$\boxed{M^{(i)}_{AB} = \frac{v}{\sqrt{2}}\, f^{(i)}_{AB}}, \qquad i = e, u, d$$

The fermion mass matrix need not be symmetric nor hermitian. However, $\exists$ unitary matrices S and T such that

$$\boxed{S^{+} M\, T = M_d}$$

$$\uparrow \text{diagonal, positive eigenvalues}$$

Proof! The matrix MM^{+} is hermitian and positive $\rightarrow$ can be diagonalized by a unitary matrix S, i.e.

$$S\,(MM^{+})\,S = M_d^2, \qquad M_d^2 = \begin{pmatrix} m_1^2 & & \\ & m_2^2 & \\ & & m_3^2 \end{pmatrix}$$

Define $M_d = \begin{pmatrix} m_1 & & \\ & m_2 & \\ & & m_3 \end{pmatrix}$, $\quad m_i > 0.$

Define $H = S M_d S^+$, $H^+ = H$

Define $V = H^{-1} M \rightarrow V^+ = M^+ H^{-1}$, $H = M V^{-1}$

V is unitary:
$$\begin{aligned} VV^+ &= H^{-1} M M^+ H^{-1} \\ &= H^{-1} S M_d^2 S^+ H^{-1} \\ &= H^{-1} S M_d S^+ S M_d S^+ H^{-1} \\ &= H^{-1} \quad H \quad\quad H \quad\quad H^{-1} \quad = 1 \end{aligned}$$

We have,
$$S^+ H S = S^+ S M_d S^+ S = M_d$$
$$= S^+ M V^+ S$$

Define $T = V^+ S$, $TT^+ = 1$.

We have proven that M is diagonalized by a biunitary transformation, $M \rightarrow S^+ M T = M_d$. ■

The mass terms are of the form
$$\bar{\Psi}_L' M \Psi_R' = \bar{\Psi}_L' S (S^+ M T) T^+ \Psi_R' \equiv \bar{\Psi}_L M_d \Psi_R ,$$

gauge eigenstates

Mass eigenstates

$\Psi_L' = S \Psi_L$

$\Psi_R' = T \Psi_R$

There are independent matrices S and T for the charged leptons and the u and d-type quarks.

The CKM mixing matrix – Cabibbo, Kobayashi, Maskawa

The charged weak current interactions for the quarks take the form $\mathcal{L} \supset \frac{g}{\sqrt{2}} J_q^{+\mu} W_\mu^+ + h.c.$ where

$$J_q^{+\mu} = \bar{u}'_{LA} \gamma^\mu d'_{LA}$$
$$= \bar{u}_{LA} \gamma^\mu \left(S_{(u)}^\dagger S_{(d)}\right)_{AB} d_{LB} \equiv \bar{u}_{LA} \gamma^\mu V_{AB} d_{LB}$$

where $V \equiv S_{(u)}^\dagger S_{(d)}$ is the CKM matrix.

V is unitary. In components, $V = \begin{pmatrix} V_{ud} & V_{us} & V_{ub} \\ V_{cd} & V_{cs} & V_{cb} \\ V_{td} & V_{ts} & V_{tb} \end{pmatrix}$

$$\mathcal{L} \supset \frac{g}{\sqrt{2}} (\bar{u}, \bar{c}, \bar{t})_L \gamma^\mu V \begin{pmatrix} d \\ s \\ b \end{pmatrix}_L W_\mu^+ + h.c.$$

Mixing angles and phases: The CKM matrix can be parametrized by a number of angles and phases. The phases are important because they lead to CP violation in the weak interactions.

Consider an $n \times n$ unitary matrix, generalizing the CKM matrix to the case with n doublets of quarks.

$n \times n$ matrix has $2n^2$ real parameters.

Unitarity = n^2 conditions → leaves n^2 real parameters.

An $n \times n$ orthogonal matrix has $\frac{n(n-1)}{2}$ parameters ($\exists \frac{n(n-1)}{2}$ planes of rotation in n-dimensions)

The difference between the unitary and orthogonal matrices is the extra phases → $n\times n$ unitary matrix has $n^2 - \frac{n(n-1)}{2}$ phases.

These phases in the CKM matrix are not all physical, because the quark fields can be rotated by phases.
Multiplying the $2n$ quark fields by phases eliminates $(2n-1)$ phases in the CKM matrix (overall multiplication of all fermions by the same phase does nothing to the CKM matrix).

Hence, there are $\boxed{n^2 - \frac{(n)(n-1)}{2} - (2n-1) \text{ physical phases}}$ in the CKM matrix, and $\boxed{\frac{n(n-1)}{2} \text{ mixing angles}}$.

2-flavors: 1 mixing angle, no phase → no CP violation

3-flavors: 3 mixing angles, 1 phase

Can write
$$\boxed{V = \begin{pmatrix} c_1 & s_1 c_3 & s_1 s_3 \\ -s_1 c_1 & c_1 c_2 c_3 - s_2 s_3 e^{i\delta} & c_1 c_2 s_3 + s_2 c_3 e^{i\delta} \\ -s_1 s_2 & c_1 s_2 c_3 + c_2 s_3 e^{i\delta} & c_1 s_2 s_3 - c_2 c_3 e^{i\delta} \end{pmatrix}}$$

– Kobayashi-Maskawa parametrization (1973)

Notation: $c_i = \cos\theta_i$, $s_i = \sin\theta_i$

Can chose $0 \le \theta_i \le \frac{\pi}{2}$, $-\pi \le \delta \le \pi$