Spontaneous Symmetry Breaking (SSB)
(Cheng + Li ch. 5.3)

Normally, symmetries lead to degeneracy between states that form multiplets of the symmetry group. The states in a multiplet transform in irreducible representations of the symmetry group.

Suppose $U = \exp(i\theta^a Q^a)$ is an element of the symmetry group, i.e. $U H U^\dagger = H$.
Hamiltonian

Suppose $H|A\rangle = E_A|A\rangle$.
Define $|B\rangle \equiv U|A\rangle$.
Then $H|B\rangle = UHU^\dagger|B\rangle = UH|A\rangle = E_A U|A\rangle = E_A|B\rangle$.
Hence, $|B\rangle$ is an eigenstate of H w/ eigenvalue E_A.

Suppose ϕ_A and ϕ_B are two fields related by the symmetry transformation, $U\phi_A U^\dagger = \phi_B$.

vacuum

If $|A\rangle = \phi_A|0\rangle$

$$|B\rangle = \phi_B|0\rangle = U\phi_A U^\dagger|0\rangle$$
$$= U|A\rangle \quad \underline{\text{if}} \quad U^\dagger|0\rangle = |0\rangle$$

However, if $U^\dagger|0\rangle \neq |0\rangle$, the states $|A\rangle$ and $|B\rangle$ defined above are generally nondegenerate.
In this case the vacuum is not invariant under the symmetry transformation, and we say that the symmetry is <u>spontaneously broken</u>.

Example: Ferromagnetism near the Curie temperature T_c

For $T > T_c$ spins are randomly aligned
$\rightarrow$ magnetization vanishes
For $T < T_c$ spins align
$\rightarrow$ $\exists$ net magnetization

Landau-Ginzburg model: Near T_c, free energy is a rotation-invariant function of the magnetization $\vec{M}$.

$$U(\vec{M}) = (\partial_i \vec{M}) \cdot (\partial_i \vec{M}) + V(\vec{M})$$

$$V(\vec{M}) = \alpha_1(T)\, \vec{M} \cdot \vec{M} + \alpha_2\, (\vec{M} \cdot \vec{M})^2 \quad , \alpha_2 > 0$$

Ground state: $\frac{\partial V}{\partial M_i} = 0 \Rightarrow \vec{M}\,(\alpha_1 + 2\alpha_2\, \vec{M} \cdot \vec{M}) = 0$

$T > T_c$: $\alpha_1(T) > 0$, min at $\vec{M} = 0$.
$T < T_c$: $\alpha_1(T) < 0$, min at $|\vec{M}| = \left(\frac{-\alpha_1}{2\alpha_2}\right)^{1/2}$

For $T < T_c$ $|\vec{M}| \neq 0$, but the direction is random
$\rightarrow$ decides which ground state

Symmetry breaking = noninvariance of vacuum
$\rightarrow$ nonvanishing order parameter.

Goldstone's Theorem

Given a continuous symmetry, Noether's theorem implies $\exists$ current $J^\mu(x)$ s.t. $\partial_\mu J^\mu = 0$.

If $\int d^3x\, J^0(x) \equiv Q$ is well defined, then it is conserved:

$$\boxed{0} = \int d^3x\, \partial_\mu J^\mu = \int d^3x\, \partial_0 J^0 + \int d^3x\, \nabla\cdot\vec{J}$$
$$\boxed{= \frac{d}{dt} Q} + \underbrace{\int d\vec{S}\cdot\vec{J}}_{=0 \text{ if } \vec{J} \text{ falls quickly enough @ } \infty}$$

Charges generate symmetry transformations: $U = \exp(i\theta^a Q^a)$.

Example: Momentum generates translations

$$\phi(x) = \exp(iP\cdot x)\,\phi(0)\,\exp(-iP\cdot x)$$

If the symmetry U is spontaneously broken, then $U|0\rangle \neq |0\rangle$.

Infinitessimal form: $(1 + i\theta^a Q^a)|0\rangle \neq |0\rangle$

$$\rightarrow \boxed{Q^a|0\rangle \neq 0}$$

Then
$$\langle 0|Q^2(t)|0\rangle = \int d^3x\, \langle 0|J_0(\vec{x},t)\,Q(t)|0\rangle$$
$$= \int d^3x\, \langle 0|e^{iP\cdot x} J_0(0) e^{-iP\cdot x} e^{iP\cdot x} Q(0) e^{-iP\cdot x}|0\rangle$$
$$= \int d^3x\, \langle 0|J_0(0)Q(0)|0\rangle$$
$$= \infty \quad \text{since the integrand is indep. of } \vec{x}.$$

However, commutators w/ Q are well defined.

Under an infinitessimal symmetry transformation,

$$\phi(x) \to \exp(i\theta Q)\,\phi(x)\,\exp(-i\theta Q)$$
$$\approx \phi(x) + i\theta\,[Q, \phi(x)]$$

If $\exp(-i\theta Q)|0\rangle = |0\rangle$, i.e. the symmetry is unbroken, then $\langle 0|[Q, \phi(x)]|0\rangle = 0$, and under a symmetry transformation $\langle 0|\phi(x)|0\rangle \to \langle 0|\phi(x)|0\rangle$

A well-defined definition of spontaneous symmetry breaking is the condition $\boxed{\langle 0|[Q(t), \phi(0)]|0\rangle \neq 0,}$ for some field ϕ.

This quantity is time independent:

$$0 = \int d^3x\,[\partial_\mu J^\mu(x), \phi(0)]$$
$$= \frac{d}{dt}\int d^3x\,[J^0(x), \phi(0)] + \underbrace{\cancel{\int d\vec{S}\cdot[\vec{J}(x), \phi(0)]}}_{\to 0}$$
$$= \frac{d}{dt}\,[Q(t), \phi(0)].$$

Hence, the condition for SSB is $\langle 0|[Q(t), \phi(0)]|0\rangle \equiv \eta \neq 0$

Insert a complete set of states s.t. $\sum_n |n\rangle\langle n| = 1$.

$$\eta = \sum_n \int d^3x\,\Big(\langle 0|J^0(x)|n\rangle\langle n|\phi(0)|0\rangle - \langle 0|\phi(0)|n\rangle\langle n|J^0(x)|0\rangle\Big)$$

$$= \sum_n \int d^3x\,\Big(e^{-ip_n\cdot x}\langle 0|J^0(0)|n\rangle\langle n|\phi(0)|0\rangle - e^{ip_n\cdot x}\langle 0|\phi(0)|n\rangle\langle n|J^0(0)|0\rangle\Big)$$

$$= \sum_n (2\pi)^3\,\delta^3(\vec{p}_n)\Big(e^{-i\omega_n t}\langle 0|J^0(0)|n\rangle\langle n|\phi(0)|0\rangle - e^{i\omega_n t}\langle 0|\phi(0)|n\rangle\langle n|J^0(0)|0\rangle\Big)$$

The terms in the sum w/ $w_n \neq 0$ would not be time independent, so they must vanish.
The term(s) with $w_n = 0$, $\vec{p}_n = 0$ must contribute to the sum because $\eta \neq 0$ by assumption.

Hence, $\exists$ a state $|n\rangle$ with $w_n = 0$, $\vec{p}_n = 0$.
If the one-particle states satisfy the relativistic dispersion relation $w_n^2 = \vec{p}_n^{\,2} + m_n^2$, then $|n\rangle$ is massless, $m_n = 0$.

This is <u>Goldstone's theorem</u>!
$\forall$ spontaneously broken continuous global symmetry, $\exists$ a massless <u>Goldstone boson</u> in the spectrum.

The Higgs Mechanism: Spontaneously broken gauge invariance.
(Cheng+Li ch. 8.3)

Unbroken global symmetries lead to degeneracies between states, selection rules and relations between scattering amplitudes.

Broken continuous global symmetries lead to massless particles.

Broken gauge invariance leads to massive vector fields. This is the Higgs mechanism, and in the Standard Model is responsible for the masses of the W and Z bosons.

Abelian Example: $\mathcal{L} = (D_\mu \phi)^\dagger (D^\mu \phi) + \mu^2 \phi^\dagger \phi - \lambda (\phi^\dagger \phi)^2 - \frac{1}{4} F_{\mu\nu} F^{\mu\nu}$

$D_\mu \phi = (\partial_\mu - i g A_\mu)\phi$

$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$

Gauge invariance: $\phi(x) \to e^{-i\alpha(x)} \phi(x)$

$$A_\mu(x) \to A_\mu(x) - \frac{1}{g} \partial_\mu \alpha(x)$$

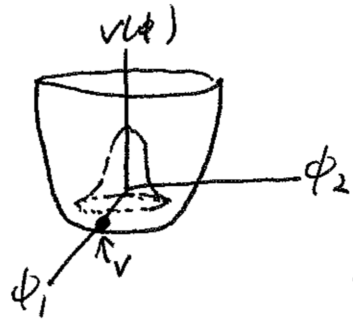
$\mu^2 > 0$: Minimum of potential $V(\phi) = -\mu^2 \phi^\dagger \phi + \lambda (\phi^\dagger \phi)^2$ is at $|\phi| = v/\sqrt{2}$, $v = (\mu^2/\lambda)^{1/2}$.

Hence, $|\langle 0 | \phi(x) | 0 \rangle| = v/\sqrt{2}$.

The vacuum expectation value (VEV) of $\phi(x)$ breaks the gauge invariance because under a gauge transformation

$\langle 0 | \phi(x) | 0 \rangle \to \langle 0 | \phi(x) | 0 \rangle \, e^{-i\alpha(x)}$

We can choose $\langle 0 | \phi(x) | 0 \rangle$ to be real, $\langle 0 | \phi(x) | 0 \rangle = v/\sqrt{2}$.



Define $\phi(x) = \frac{1}{\sqrt{2}}(\phi_1(x) + i\,\phi_2(x))$,
ϕ_1 and ϕ_2 real.

Define $\phi_1' = \phi_1 - v$
$\phi_2' = \phi_2$

The potential is $V(\phi) = -\mu^2 \phi^\dagger \phi + \lambda(\phi^\dagger\phi)^2$

$$= -\frac{\mu^2}{2}(\phi_1^2 + \phi_2^2) + \frac{\lambda}{4}(\phi_1^2+\phi_2^2)^2$$

$$= -\frac{\mu^2}{2}\left((\phi_1'+v)^2 + \phi_2'^2\right) + \frac{\lambda}{4}\left((\phi_1'+v)^2 + \phi_2'^2\right)^2$$

$$= -\frac{\mu^4}{4\lambda} + \mu^2\phi_1'^2 + \lambda v\,\phi_1'(\phi_1'^2+\phi_2'^2) + \frac{\lambda}{4}(\phi_1'^2+\phi_2'^2)^2$$

Note that ϕ_1' has mass $\sqrt{2}\mu$, but ϕ_2' would be massless. If this were the end of the story, ϕ_2' would be the Goldstone boson of the spontaneously broken $U(1)$. However, $\phi_2'(x)$ can be transformed away by a gauge transformation, so we have to work harder to identify the physical degrees of freedom.

Consider the kinetic term for ϕ:

$$|D_\mu\phi|^2 = |(\partial_\mu - igA_\mu)\phi|^2$$

$$= \frac{1}{2}\left[(\partial_\mu - igA_\mu)(v+\phi_1'+i\phi_2')\right]\left[(\partial_\mu + igA_\mu)(v+\phi_1'-i\phi_2')\right]$$

$$= \frac{1}{2}(\partial_\mu\phi_1' + gA_\mu\phi_2')^2 + \frac{1}{2}(\partial_\mu\phi_2' - gA_\mu\phi_1')^2$$
$$- gvA^\mu(\partial_\mu\phi_2' + gA_\mu\phi_1') + \underbrace{\frac{g^2v^2}{2}A_\mu A^\mu}_{\text{Mass term for } A^\mu \text{ !!!}}$$

The term $gvA^\mu \partial_\mu \phi_2'$ mixes A^μ with the would-be Goldstone boson ϕ_2'. To make the physical degrees of freedom clear we gauge this term away.

Define $\phi(x) = \frac{1}{\sqrt{2}}(v + \eta(x)) \exp(i\xi(x)/v)$

$$= \frac{1}{\sqrt{2}}(v + \eta(x) + i\,\xi(x) + \ldots)$$

like $\phi_1'(x)$ — like $\phi_2'(x)$

Define the gauge-transformed fields (Unitary Gauge)

$$\phi^u(x) \equiv \exp(-i\xi(x)/v)\,\phi(x) = \frac{1}{\sqrt{2}}(v + \eta(x))$$
$$B_\mu(x) \equiv A_\mu(x) - \frac{1}{gv}\partial_\mu \xi(x)$$

The covariant derivative $D_\mu \phi$ transforms like ϕ, so

$$D_\mu \phi = \exp(i\xi(x)/v)(\partial_\mu \phi^u - ig B_\mu \phi^u)$$

$$= \exp(i\xi(x)/v)(\partial_\mu \eta(x) - ig B_\mu (v + \eta(x)))/\sqrt{2}$$

$$|D_\mu \phi|^2 = \frac{1}{2}|\partial_\mu \eta - ig B_\mu (v+\eta)|^2$$

Also, $F_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu$.

The Lagrangian is
$$\mathcal{L} = \frac{1}{2}|\partial_\mu \eta - ig B_\mu (v+\eta)|^2 + \frac{\mu^2}{2}(v+\eta)^2 - \frac{\lambda}{4}(v+\eta)^4 - \frac{1}{4}(\partial_\mu B_\nu - \partial_\nu B_\mu)^2$$

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \eta)^2 - \mu^2 \eta^2 - \frac{1}{4}(\partial_\mu B_\nu - \partial_\nu B_\mu)^2 + \frac{1}{2}(gv)^2 B_\mu B^\mu$$

$$+ \frac{1}{2} g^2 B_\mu B^\mu \eta (2v + \eta) - \lambda v^2 \eta^3 - \frac{1}{4}\lambda \eta^4$$

Note that the field $\xi(x)$ has completely disappeared from the Lagrangian.
However, the field B_μ has mass gv.

It is said that the gauge field has eaten the would-be Goldstone boson to become massive.

Note that the number of propagating degrees of freedom has not changed. A massless vector field has 2 propagating degrees of freedom (helicity ± 1). A massive vector field has an additional helicity 0 ("longitudinal") mode.