Ward Identities and Anomalies, Cheng + Li 6.1, 6.2

Ward identities are the consequence of symmetries in quantum field theory.

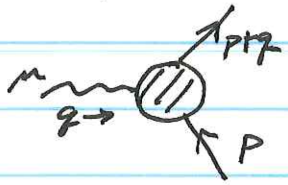
Consider the theory of a complex scalar field ϕ with a $U(1)$ symmetry $\phi \to e^{i\theta}\phi$.
As a consequence there is a conserved current J^μ such that $\partial_\mu J^\mu = 0$. If $\mathcal{L} = (\partial_\mu \phi)^\dagger(\partial^\mu \phi) + f(\phi^\dagger \phi)$, then $J_\mu = i\left[(\partial_\mu \phi)^\dagger \phi - (\partial_\mu \phi)\phi^\dagger\right]$.

Using the equal-time commutator $[\partial_0 \phi^\dagger(\vec{x},t), \phi(\vec{x}',t)] = -i\delta^3(\vec{x}-\vec{x}')$ we also have the commutators

$$[J_0(\vec{x},t), \phi(\vec{x}',t)] = i\left[(\partial_0\phi)^\dagger_{(\vec{x},t)}, \phi(\vec{x}',t)\right]\phi(\vec{x},t)$$
$$= \delta^3(\vec{x}-\vec{x}')\,\phi(\vec{x},t)$$

$$[J_0(\vec{x},t), \phi^\dagger(\vec{x}',t)] = -\delta^3(\vec{x}-\vec{x}')\,\phi^\dagger(\vec{x},t)$$

Consider the 3-pt function



$$\Gamma_\mu(p,q) = \int d^4x\, d^4y\, e^{-i(q\cdot x + p\cdot y)} \langle 0|T(J_\mu(x)\phi(y)\phi^\dagger(0))|0\rangle$$

$$q^\mu \Gamma_\mu(p,q) = -i\int d^4x\, d^4y\; e^{-i(q\cdot x+p\cdot y)}\,\partial_x^\mu \langle 0|T(J_\mu(x)\phi(y)\phi^\dagger(0))|0\rangle$$

$$= -i\int d^4x\, d^4y\; e^{-i(q\cdot x+p\cdot y)}\Big[\langle 0|T(\partial^\mu J_\mu(x)\,\phi(y)\,\phi^\dagger(0))|0\rangle$$

$$+\,\delta(x^0-y^0)\langle 0|T([J_0(x),\phi(y)]\phi^\dagger(0))|0\rangle + \delta(x_0)\langle 0|T([J_0(x),\phi^\dagger(0)]\phi(y))|0\rangle\Big]$$

$$\boxed{q^\mu \Gamma_\mu (p,q)} = -i \int d^4x\, e^{-i(p+q)\cdot x} \langle 0 | T[\phi(x)\phi^\dagger(0)] | 0 \rangle$$

$$+ i \int d^4y\, e^{-ip\cdot y} \langle 0 | T[\phi^\dagger(0)\, \phi(y)] | 0 \rangle$$

$$\boxed{= -i \left(\Delta(p+q) - \Delta(p) \right)} \qquad \text{Ward identity}$$

where $\Delta(p)$ is the propagator $\Delta(p) = \int d^4x\, e^{-ip\cdot x} \langle 0 | T(\phi(x)\phi^\dagger(0)) | 0 \rangle$

Now consider a theory of fermions. Define the vector, axial vector and pseudoscalar currents:

$$V_\mu(x) = \bar\Psi(x) \gamma_\mu \Psi(x)$$
$$A_\mu(x) = \bar\Psi(x) \gamma_\mu \gamma_5 \Psi(x)$$
$$P(x) = \bar\Psi(x) \gamma_5 \Psi(x)$$

If $\mathcal{L} = \bar\Psi(i\partial\!\!\!/ - m)\Psi$, then from the equations of motion

$$\partial^\mu V_\mu = 0$$
$$\partial^\mu A_\mu = 2im P$$

Use $\partial_x^\mu T(J_\mu(x)\, \mathcal{O}(y)) = T(\partial^\mu J_\mu(x)\, \mathcal{O}(y)) + [J_0(x), \mathcal{O}(y)]\, \delta(x^0 - y^0)$

and $[V_0(x), A_0(y)]\, \delta(x^0 - y^0) = 0$

Define $T_{\mu\nu\lambda}(k_1, k_2, q) = i \int d^4x\, d^4y\, \langle 0 | T(V_\mu(x) V_\nu(y) A_\lambda(0)) | 0 \rangle\, e^{i(k_1\cdot x + k_2\cdot y)}$

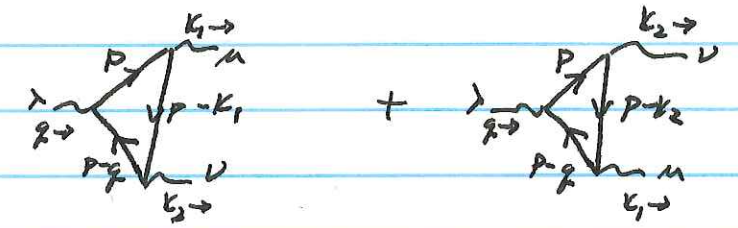
$$T_{\mu\nu}(k_1, k_2, q) = i \int d^4x\, d^4y\, \langle 0 | T(V_\mu(x) V_\nu(y) P(0)) | 0 \rangle\, e^{i(k_1\cdot x + k_2\cdot y)}$$

Exercise. $\Rightarrow$

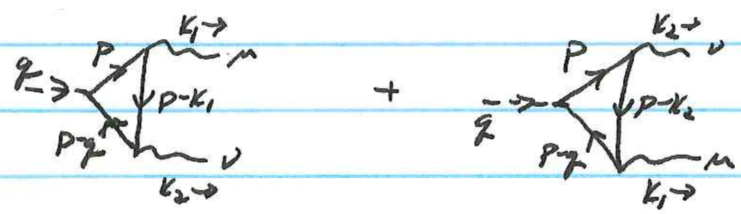
$$\boxed{\begin{gathered} k_1^\mu T_{\mu\nu\lambda} = k_2^\nu T_{\mu\nu\lambda} = 0 \\ q^\lambda T_{\mu\nu\lambda} = 2m\, T_{\mu\nu} \end{gathered}} \qquad \text{Ward identities.}$$

We assumed the classical current conservation relations in the derivation of the Ward identities. In chiral theories <u>renormalization</u> may violate these relations.

<u>Triangle Anomaly</u>:



$T_{\mu\nu\lambda}$



$T_{\mu\nu}$

$$T_{\mu\nu\lambda} = i\int \frac{d^4p}{(2\pi)^4}(-1)\,\mathrm{Tr}\left[\frac{i}{\not p - m + i\epsilon}\gamma_\lambda\gamma_5 \frac{i}{(\not p - \not q) - m + i\epsilon}\gamma_\nu \frac{i}{(\not p - \not k_1) - m + i\epsilon}\gamma_\mu\right]$$

$$+ \begin{pmatrix} k_1 \leftrightarrow k_2 \\ \mu \leftrightarrow \nu \end{pmatrix}$$

$$T_{\mu\nu} = i\int \frac{d^4p}{(2\pi)^4}(-1)\,\mathrm{Tr}\left[\frac{i}{\not p - m + i\epsilon}\gamma_5 \frac{i}{(\not p - \not q) - m + i\epsilon}\gamma_\nu \frac{i}{(\not p - \not k_1) - m + i\epsilon}\gamma_\mu\right]$$

$$+ \begin{pmatrix} k_1 \leftrightarrow k_2 \\ \mu \leftrightarrow \nu \end{pmatrix}$$

Use $\not q\gamma_5 = \gamma_5(\not p - \not q - m) + (\not p - m)\gamma_5 + 2m\gamma_5$

$$\Rightarrow \quad q^\lambda T_{\mu\nu\lambda} = 2m T_{\mu\nu} + \Delta^{(1)}_{\mu\nu} + \Delta^{(2)}_{\mu\nu}$$

$$\Delta^{(1)}_{\mu\nu} = \int \frac{d^4p}{(2\pi)^4} \mathrm{Tr}\left[\frac{i}{\not p - m + i\epsilon} \gamma_5 \gamma_\nu \frac{i}{(\not p - \not k_1) - m + i\epsilon} \gamma_\mu \right.$$

$$\left. - \frac{i}{(\not p - \not k_2) - m + i\epsilon} \gamma_5 \gamma_\nu \frac{i}{(\not p - \not q) - m + i\epsilon} \gamma_\mu \right]$$

$$\Delta^{(2)}_{\mu\nu} = \int \frac{d^4p}{(2\pi)^4} \mathrm{Tr}\left[\frac{i}{\not p - m + i\epsilon} \gamma_5 \gamma_\mu \frac{i}{(\not p - \not k_2) - m + i\epsilon} \gamma_\nu \right.$$

$$\left. - \frac{i}{(\not p - \not k_1) - m + i\epsilon} \gamma_5 \gamma_\mu \frac{i}{(\not p - \not q) - m + i\epsilon} \gamma_\nu \right]$$

The Ward identity is satisfied if $\Delta^{(1)}_{\mu\nu} + \Delta^{(2)}_{\mu\nu} = 0$.

Shifting the integral in the second term of $\Delta^{(1)}_{\mu\nu}$, $p \to p + k_2$, then using $k_2 - q = k_1$, the two integrals in $\Delta^{(1)}_{\mu\nu}$ would cancel.
Similarly the two integrals in $\Delta^{(2)}_{\mu\nu}$ would cancel shifting $p \to p + k_1$ in the second integral.

☆ However, these integrals are linearly divergent and we cannot shift the integration variable without changing the integral. As a result we will find $\Delta^{(1)}_{\mu\nu} + \Delta^{(2)}_{\mu\nu} \neq 0$, and the Ward identity is not satisfied.

Linearly divergent integrals: $\Delta(a) \equiv \int_{-\infty}^{\infty} dx \left[f(x+a) - f(x) \right]$

Expand: $\Delta(a) = \int_{-\infty}^{\infty} dx \left[a f'(x) + \frac{a^2}{2} f''(x) + \dots \right]$

$$= a \left[f(\infty) - f(-\infty) \right] + \frac{a^2}{2} \left[f'(\infty) - f'(-\infty) \right] + \dots$$

If $\int_{-\infty}^{\infty} f(x)\,dx$ converges or diverges at most logarithmically, then $0 = f(\pm\infty) = f'(\pm\infty) = \dots \rightarrow \Delta(a) = 0$.

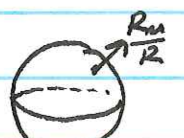
For a linearly divergent integral, $0 \neq f(\pm\infty)$
$0 = f'(\pm\infty) = \dots$

$$\rightarrow \Delta(a) = a \left[f(\infty) - f(-\infty) \right]$$

In n dimensions, $\Delta(a) \equiv \int d^n r \left[f(r+a) - f(r) \right]$

$$\Delta(a) = \int d^n r \left[a^\mu \partial_\mu f(r) + \tfrac{1}{2} a^\mu \partial_\mu a^\nu \partial_\nu f(r) + \dots \right]$$

$$= a^\mu \frac{R_\mu}{R} f(R)\, S_n(R)$$



surface area of n-dim'l hyper-sphere w/ radius R.

Minkowski $n=4$: $\Delta(a) = a^\mu \int d^4 r\, \partial_\mu f(r) = 2 i \pi^2 a^\mu \lim_{R\to\infty} R^2 R_\mu f(R)$

Our calculation of $T_{\mu\nu\lambda}$ is ambiguous because it depends on the definition of the loop momentum. It is not invariant under shifts of the momentum integration variable.

We can take $p \to p + \underbrace{\alpha k_1 + (\alpha - \beta) k_2}_{a}$

$$\Delta_{\mu\nu\lambda}(a) \equiv T_{\mu\nu\lambda}(a) - T_{\mu\nu\lambda}(0)$$

$$= (-1)\int \frac{d^4 p}{(2\pi)^4} \left(Tr\left[\frac{1}{\not p + \not a - m + i\epsilon} \gamma_\lambda \gamma_5 \frac{1}{\not p + \not a + \not q - m + i\epsilon} \gamma_\nu \frac{1}{(\not p + \not a - \not k_1) - m + i\epsilon} \gamma_\mu \right] \right.$$

$$\left. - Tr\left[\frac{1}{\not p - m + i\epsilon} \gamma_\lambda \gamma_5 \frac{1}{\not p + \not q - m + i\epsilon} \gamma_\nu \frac{1}{\not p - \not k_1 - m + i\epsilon} \gamma_\mu \right] \right)$$

$$+ \begin{pmatrix} k_1 \leftrightarrow k_2 \\ \mu \leftrightarrow \nu \end{pmatrix}$$

$$\equiv \Delta^{(1)}_{\mu\nu\lambda} + \Delta^{(2)}_{\mu\nu\lambda}$$

$$\Delta^{(1)}_{\mu\nu\lambda} = (-1)\int \frac{d^4 p}{(2\pi)^4} a^\sigma \frac{\partial}{\partial p^\sigma} Tr\left[\frac{1}{\not p - m + i\epsilon} \gamma_\lambda \gamma_5 \frac{1}{\not p \not q - m + i\epsilon} \gamma_\nu \frac{1}{\not p \not k_1 - m + i\epsilon} \gamma_\mu \right]$$

$$= \frac{-2i\pi^2 a^\sigma}{(2\pi)^4} \lim_{p \to \infty} p^2 p_\sigma \, Tr\left[\frac{\not p \gamma_\lambda \gamma_5 \not p \gamma_\nu \not p \gamma_\mu}{p^6} \right]$$

$$= \frac{2i\pi^2 a^\sigma}{(2\pi)^4} \lim_{p \to \infty} \frac{p_\sigma p^\rho}{p^2} 4i \epsilon_{\mu\nu\lambda\rho}$$

Replace $p^\sigma p^\rho \to p^2 g^{\rho\sigma}/4$

$$\boxed{\Delta^{(1)}_{\mu\nu\lambda} = \epsilon_{\rho\mu\nu\lambda} \frac{a^\rho}{8\pi^2}}$$

$\Delta^{(2)}_{\mu\nu\lambda}$ is the same as $\Delta^{(1)}_{\mu\nu\lambda}$ with the exchanges
$k_1 \leftrightarrow k_2$, $\mu \leftrightarrow \nu$.
Using $q = \alpha k_1 + (\alpha - \beta) k_2$, we get

$$\Delta_{\mu\nu\lambda} = \Delta^{(1)}_{\mu\nu\lambda} + \Delta^{(2)}_{\mu\nu\lambda} = \frac{\beta}{8\pi^2} \epsilon_{\rho\mu\nu\lambda} (k_1 - k_2)^\rho$$

Hence $T_{\mu\nu\lambda}$ is ambiguous up to the addition

$$\boxed{T_{\mu\nu\lambda} \to T_{\mu\nu\lambda} - \frac{\beta}{8\pi^2} \epsilon_{\mu\nu\lambda\rho} (k_1 - k_2)^\rho}$$

Similarly, $\Delta^{(1)}_{\mu\nu} = \Delta^{(2)}_{\mu\nu} = -\frac{1}{8\pi^2} \epsilon_{\mu\nu\sigma\rho} k_1^\sigma k_2^\rho$

$$\Rightarrow \boxed{q^\lambda T_{\mu\nu\lambda} = 2m T_{\mu\nu} - \frac{(1-\beta)}{4\pi^2} \epsilon_{\mu\nu\sigma\rho} k_1^\sigma k_2^\rho}$$

where β is so far undetermined

Similarly, $\boxed{k_1^\mu T_{\mu\nu\lambda} = \frac{(1+\beta)}{8\pi^2} \epsilon_{\nu\lambda\sigma\rho} k_1^\sigma k_2^\rho}$ (Exercise)

The terms $\propto (1-\beta)$ and $(1+\beta)$ are the anomalous terms. It is not possible to simultaneously set both of these terms to zero.

To maintain the vector Ward identity we chose $\boxed{\beta = -1}$

$$\Rightarrow \boxed{q^\lambda T_{\mu\nu\lambda} = 2m T_{\mu\nu} - \frac{1}{2\pi^2} \epsilon_{\mu\nu\sigma\rho} k_1^\sigma k_2^\rho}$$

The anomalous term corresponds to a modification of the classical current divergence equation:

$$\partial^\mu A_\mu = 2im\,P + \underbrace{\frac{1}{(4\pi)^2}\,\epsilon^{\mu\nu\rho\sigma} F_{\mu\nu} F_{\rho\sigma}}_{\text{The ABJ anomaly}}$$

Field strengths for gauged vector symmetry ~ electromagnetism.

↑ Adler, Bell, Jackiw (1969)

Non-Abelian anomalies:

$$T^{abc}_{\mu\nu\lambda}(k_1,k_2,q) = i\int d^4x\, d^4y\, \langle 0|T\left(V_\mu^a(x)\, V_\nu^b(y)\, A_\lambda^c(0)\right)|0\rangle\, e^{i(k_1\cdot x + k_2\cdot y)}$$

where

$$V_\mu^a = \bar\Psi T^a \gamma_\mu \psi$$

$$A_\lambda^c = \bar\Psi T^c \gamma_\lambda \gamma_5 \psi$$

Also define

$$P^c = \bar\Psi T^c \gamma_5 \psi$$

Then $$q^\lambda T^{abc}_{\mu\nu\lambda} = 2m\, T^{abc}_{\mu\nu} + \text{commutators} \underbrace{- \frac{1}{2\pi^2}\,\epsilon_{\mu\nu\rho\sigma} k_1^\rho k_2^\sigma D^{abc}}_{\text{ABJ Anomaly}}$$

where $$\boxed{D^{abc} = \frac{1}{2}\,\mathrm{Tr}\left(\{T^a, T^b\}\, T^c\right)}$$

Gauge anomalies must vanish for renormalizability.
Hence, if T^a, T^b, T^c are generators of gauge invariances then summed over representations of chiral fermions, $D^{abc} = 0$.
For $U(1)$ gauge invariance, we need $\boxed{\sum Q_i^3 = 0}$ where Q_i = fermion charge.
To couple to gravity, we can replace two T^a's w/ $T_{\mu\nu}$ - stress tensor → $\boxed{\sum Q_i = 0}$