# Symmetries and the Functional Integral

One loose end that we need to tie up is a derivation of the Ward-Takahashi Identity (Ward ID for short) that was so important in our discussions of the renormalization of QED.

The functional integral formalism provides an easy way to relate symmetries to conservation laws.

## Quantum Equations of Motion:

Consider the theory of a scalar field with some Lagrangian $\mathcal{L}(\phi)$. Consider the n-point function,

$$\langle 0|T\phi(x_1)\cdots\phi(x_n)|0\rangle = Z^{-1}\int\mathcal{D}\phi\, \exp\left(i\int d^4x\,\mathcal{L}\right)\phi(x_1)\cdots\phi(x_n).$$

Under a change of variables $\phi(x)\to\phi'(x)=\phi(x)+\epsilon(x)$, the measure $\mathcal{D}\phi$ is invariant, i.e. $\mathcal{D}\phi=\mathcal{D}\phi'$. The functional integral is also invariant under a change of variables.

$$\int\mathcal{D}\phi\, e^{iS[\phi]}\phi(x_1)\cdots\phi(x_n) = \int\mathcal{D}\phi'\, e^{iS[\phi']}\phi'(x_1)\cdots\phi'(x_n)$$
$$=\int\mathcal{D}\phi\, e^{iS[\phi']}\phi'(x_1)\cdots\phi'(x_n)$$

Expanding to linear order in $\epsilon(x)$, we get

$$0=\int\mathcal{D}\phi\, e^{iS[\phi]}\Big\{ i\int d^4x\,\epsilon(x)\frac{\delta}{\delta\phi(x)}\left(\int d^4x'\mathcal{L}\right)\cdot\phi(x_1)\cdots\phi(x_n)$$
$$+\epsilon(x_1)\phi(x_2)\cdots\phi(x_n)+\cdots+\phi(x_1)\cdots\phi(x_{n-1})\epsilon(x_n)\Big\}$$

The functional derivative of the Lagrangian gives the equation of motion operator:

$$\frac{\delta S}{\delta \phi} = \frac{\delta}{\delta \phi(x)} \int d^4x' \, \mathcal{L}\left(\phi(x'), \partial_\mu \phi(x')\right)$$

$$= \int d^4x' \left\{ \delta(x-x') \frac{\partial \mathcal{L}}{\partial \phi(x')} + \left(\partial_\mu \delta(x-x')\right) \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi(x'))} \right\}$$

$$= \frac{\partial \mathcal{L}}{\partial \phi(x)} - \partial_\mu \left( \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi(x))} \right)$$

We can write,

$$0 = \int \mathcal{D}\phi \; e^{iS} \; i \int d^4x \, \epsilon(x) \left\{ \frac{\delta S}{\delta \phi(x)} \phi(x_1) \cdots \phi(x_n) \right.$$
$$\left. - i\delta^4(x-x_1)\phi(x_2) \cdots \phi(x_n) - \cdots - i\phi(x_1) \cdots \phi(x_{n-1})\delta^4(x-x_n) \right\}$$

This identity is valid for any infinitessimal $\epsilon(x)$, so we get the relation

$$\boxed{\left\langle \frac{\delta S}{\delta \phi(x)} \phi(x_1) \cdots \phi(x_n) \right\rangle = \sum_{i=1}^{n} \left\langle \phi(x_1) \cdots \left(i\delta^4(x-x_i)\right) \cdots \phi(x_n) \right\rangle},$$

where $\langle \; \rangle$ denotes the time-ordered correlation fn. with derivatives on $\phi(x)$ pulled out of the correlator. (The derivatives are pulled out of the functional integral when the $\int d^4x \, \epsilon(x)$ is pulled out.)

These are the quantum generalizations of the eqs. of motion and are called the <u>Schwinger-Dyson equations</u>.

The Schwinger-Dyson equations imply that the eqs. of motion are satisfied by $\phi(x)$ in all Green's functions up to contact terms arising from the nontrivial commutation relations of the field operators.

Example: The Feynman propagator as a Green's function

Suppose $\mathcal{L} = \frac{1}{2}(\partial_\mu \phi)^2 - \frac{1}{2} m^2 \phi^2$.
The Schwinger-Dyson eqn. arising from variation of the 1pt fn. under a field redefinition is:

$$\left\langle \frac{\delta S}{\delta \phi(x)} \phi(x_1) \right\rangle = \left\langle i \delta^4(x - x_1) \right\rangle$$

$$-(\partial_\mu \partial^\mu + m^2) \underbrace{\langle 0 | T \phi(x) \phi(x_1) | 0 \rangle}_{\text{Feynman propagator}} = i \delta^4(x - x_1)$$

The Feynman propagator is a Green's function of the Klein-Gordon operator, which we have seen before.

# Quantum Conservation Laws

Classical Noether's Theorem: Symmetry of action $\rightarrow$ Conservation Law

Quantum Version: Ward Identity

Consider a symmetry transformation of the fields in an n-point function.

$$\phi_a(x) \rightarrow \phi_a(x) + \epsilon \Delta \phi_a(x)$$
$$S[\phi_a] \rightarrow S[\phi_a]$$

Constant $\epsilon$: $\mathcal{L}(\phi_a) \rightarrow \mathcal{L}(\phi_a) + \epsilon \partial_\mu F^\mu$ for some $F^\mu(\phi_a)$.

If we consider the variation of the Lagrangian when $\epsilon$ is allowed to depend on $x$, $\mathcal{L}$ picks up another term:

$$\mathcal{L}(\phi_a) \rightarrow \mathcal{L}(\phi_a) + (\partial_\mu \epsilon) \Delta \phi_a \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_a)} + \epsilon \partial_\mu F^\mu.$$

Then $\frac{\delta}{\delta \epsilon(x)} \int d^4x \, \mathcal{L}(\phi_a + \epsilon(x) \Delta \phi_a) = -\partial_\mu J^\mu$,

where $J^\mu$ is the Noether current,

$$\boxed{J^\mu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_a)} \Delta \phi_a - F^\mu}$$

Now consider the n-point function:

$$\frac{1}{Z}\int \mathcal{D}\phi_a\, e^{iS[\phi_a]}\, \phi_{a_1}(x_1)\cdots\phi_{a_n}(x_n)$$

$$= \frac{1}{Z}\int \mathcal{D}\phi_a'\, e^{iS[\phi_a']}\, \phi_{a_1}'(x_1)\cdots\phi_{a_n}'(x_n),$$

where $\phi_a'(x) = \phi_a(x) + \epsilon(x)\Delta\phi_a(x)$.
Expanding to linear order in $\epsilon(x)$,

$$0 = \int \mathcal{D}\phi_a\, e^{iS[\phi_a]}\Big\{ i\int d^4x\, \epsilon(x)\frac{\delta}{\delta\epsilon(x)}\int d^4x'\, \mathcal{L}[\phi_a+\epsilon\Delta\phi_a]\phi_{a_1}(x_1)\cdots\phi_{a_n}(x_n)$$
$$+\epsilon(x_1)\Delta\phi_a(x_1)\phi_a(x_2)\cdots\phi_{a_n}(x_n)$$
$$+\cdots + \phi_{a_1}(x_1)\cdots\phi_{a_{n-1}}(x_{n-1})\epsilon(x_n)\Delta\phi_{a_n}(x_n)\Big\}$$

Writing $\epsilon(x_i)\Delta\phi_{a_i}(x_i) = \int d^4x\, \epsilon(x)\Delta\phi_{a_i}(x_i)\delta^4(x-x_i)$ and insisting that the variation of the n-pt. function vanish for any infinitessimal $\epsilon(x)$, we get

$$0 = \int \mathcal{D}\phi_a\, e^{iS[\phi_a]}\Big\{ -i\partial_\mu J^\mu(x)\phi_{a_1}(x_1)\cdots\phi_{a_n}(x_n)$$
$$+\Delta\phi_{a_1}(x_1)\phi_{a_2}(x_2)\cdots\phi_{a_n}(x_n)\delta^4(x-x_1)$$
$$+\cdots+\phi_{a_1}(x_1)\cdots\Delta\phi_{a_n}(x_n)\delta^4(x-x_n)\Big\}$$

Hence,

$$\boxed{\begin{aligned}&\partial_\mu\langle J^\mu(x)\,\phi_{a_1}(x_1)\cdots\phi_{a_n}(x_n)\rangle\\ &= -i\,\langle \Delta\phi_{a_1}(x_1)\delta^4(x-x_1)\phi_{a_2}(x_2)\cdots\phi_{a_n}(x_n)\\ &\quad +\cdots+\phi_{a_1}(x_1)\cdots\Delta\phi_{a_n}(x_n)\delta^4(x-x_n)\rangle\end{aligned}}$$

Hence, Noether's Theorem is satisfied in Green's functions up to contact terms.

## Ward Identity

Consider the global part of the gauge invariance of QED:

$$\psi(x) \to e^{i\alpha}\psi(x) \approx (1+ie\alpha)\psi(x)$$
$$\bar{\psi}(x) \to e^{-i\alpha}\bar{\psi}(x) \approx (1-ie\alpha)\bar{\psi}(x)$$
$$A_\mu \to A_\mu.$$

Now let $\alpha$ depend on $x$ (but do not transform $A_\mu(x)$ as we would for the gauge invariance).

Then
$$\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \bar{\psi}(i\partial\!\!\!/ - eA\!\!\!/)\psi$$
$$\to \mathcal{L} + \bar{\psi}i\gamma^\mu(ie\partial_\mu\alpha)\psi$$
$$= \mathcal{L} - e(\partial_\mu\alpha)\bar{\psi}\gamma^\mu\psi.$$

So we identify the current $\boxed{J^\mu = e\bar{\psi}\gamma^\mu\psi}$.

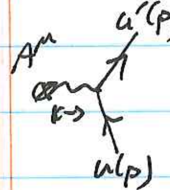
The quantum version of Noëther's theorem for the 2-pt. function is the Ward Identity:

$$\boxed{\begin{aligned} &i\partial_\mu\langle 0|T J^\mu(x)\,\psi(x_1)\,\bar{\psi}(x_2)|0\rangle \\ &= -ie\,\delta^4(x-x_1)\langle 0|T\,\psi(x_1)\bar{\psi}(x_2)|0\rangle \\ &\quad + ie\,\delta^4(x-x_2)\langle 0|T\,\psi(x_1)\,\bar{\psi}(x_2)|0\rangle \end{aligned}}$$

Ward Identity.

# Equality of Physical and Renormalized Charges

The physical electron charge is determined by the amplitude for the electron to scatter off of a static electromagnetic field:




$$-\bar{u}'(p)\, i\,\tilde{\Gamma}_\mu(p,p,0)\,u = -i\, e_{\text{physical}}\; \bar{u}'(p)\,\gamma_\mu\, u(p)$$

$\nwarrow$ physical charge, measured in expt

$k=0$ $\qquad \not{p}u = mu \qquad p^2 = m^2$

$\qquad \not{p}u' = mu'$

With our physical renormalization conditions

$$\tilde{S}^{-1}(p) = \frac{\not{p}-m}{i} + O(\not{p}-m)^2 \;;\; \tilde{\Gamma}^\mu(p,p) = \gamma^\mu e_{\text{physical}}$$

Take derivative of Ward Identity: $\qquad \tilde{\psi} \to e^{ie\alpha(x)}\tilde{\psi}$

$\swarrow$ Renormalized Charge

$$\frac{\partial}{\partial k^\nu}\left(-ik_\mu \tilde{\Gamma}^\mu(p+k,p)\right) = e\frac{\partial}{\partial k_\nu}\left(\tilde{S}^{-1}(p+k) - \tilde{S}^{-1}(p)\right)$$

$$= -i\,\tilde{\Gamma}_\nu(p+k,p) + k^\mu \frac{\partial}{\partial k^\nu}\left(-i\tilde{\Gamma}_\mu(p+k,p)\right)$$

$$= \frac{e}{i}\gamma_\nu + \text{higher order in } \not{p}-m$$

Now take $p$ to mass shell, $k \to 0$, sandwich between $\bar{u}'$, $u$:

$$-i e_{\text{physical}}\, \bar{u}'(p)\gamma_\nu u(p) = \frac{e}{i}\bar{u}'(p)\gamma_\nu u(p)$$

$$\to \boxed{e_{\text{physical}} = e}$$

Also, renorm. conditions + Ward ID as $k \to 0$, $p$ on shell $\to$ vertex renorm = field renorm, i.e. $\boxed{Z_1 = Z_2}$.