Non-Abelian Gauge Theory

QED can be constructed by beginning with the theory of a Dirac fermion with a global $U(1)$ symmetry, and introducing a gauge field $A_\mu(x)$ to lift the global symmetry to a local gauge invariance.

The free Dirac spinor field has Lagrangian

$$\mathcal{L} = \bar{\Psi}(i\partial\!\!\!/ - m)\Psi.$$

The $U(1)$ symmetry acts as $\begin{cases} \psi \to e^{ie\theta}\psi \\ \bar{\psi} \to e^{-ie\theta}\bar{\psi} \end{cases}$

If θ depends on x, then $\mathcal{L}$ is not invariant:

$$\mathcal{L} \to \mathcal{L} - e(\partial_\mu\theta)\bar{\Psi}\gamma^\mu\psi$$

To cancel the extra term we introduce a gauge field $A_\mu(x)$ and modify the Lagrangian by

$$\mathcal{L} = \bar{\Psi}(i\partial\!\!\!/ - m)\psi - eA_\mu\bar{\Psi}\gamma^\mu\psi$$

$\mathcal{L}$ is now invariant under the local transformation,

$$\begin{cases} \psi \to e^{ie\theta}\psi \\ \bar{\Psi} \to e^{-ie\theta}\bar{\Psi} \\ A_\mu \to A_\mu - \partial_\mu\theta \end{cases}$$

Notice that the gauge covariant derivative $D_\mu\psi \equiv \partial_\mu\psi + ieA_\mu\psi$ transforms the same way as what it acts on!

$$D_\mu\psi \to e^{ie\theta}D_\mu\psi$$

Starting with a Lagrangian containing a global $U(1)$ symmetry, we obtain a gauge-invariant Lagrangian by replacing derivatives with covariant derivatives, $\partial_\mu \to D_\mu$. This is the <u>minimal coupling procedure</u>.

So far the gauge field A_μ is nondynamical. To introduce dynamics for A_μ we need a gauge-invariant, Lorentz-invariant expression with derivatives of A_μ to include in $\mathcal{L}$. Of course, we know the answer — $F_{\mu\nu}F^{\mu\nu}$ satisfies the desired properties with $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$.

The field strength $F_{\mu\nu}$ appears naturally as the commutator of covariant derivatives, which explains why $F_{\mu\nu}$ is gauge-invariant.

$$[D_\mu, D_\nu]\psi \to e^{ie\theta(x)}[D_\mu, D_\nu]\psi$$

because the covariant derivative transforms as whatever it acts on. Expanding the commutator,

$$[D_\mu, D_\nu]\psi = \cancelto{0}{[\partial_\mu, \partial_\nu]}\psi + ie\left([\partial_\mu, A_\nu] - [\partial_\nu, A_\mu]\right)\psi - e^2\cancelto{0}{[A_\mu, A_\nu]}\psi$$

$$= ie\underbrace{(\partial_\mu A_\nu - \partial_\nu A_\mu)}_{F_{\mu\nu}}\psi$$

Since $\boxed{[D_\mu, D_\nu]\psi = ie F_{\mu\nu}\psi}$ stransforms as ψ, $F_{\mu\nu}$ must be gauge invariant, as is easily checked.

We recover QED as the most general renormalizable parity-conserving theory of a Dirac fermion coupled to a $U(1)$ gauge field:

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \bar{\psi}(i\not{D} - m)\psi .$$

To construct a theory invariant under local non-Abelian transformations we mimic the above steps which led to QED.

Suppose we have a theory invariant under a non-Abelian global symmetry with generators T^a, acting on a set of fields, for example a collection of n Dirac spinor fields,

$$\psi(x) = \begin{pmatrix} \psi_1(x) \\ \vdots \\ \psi_n(x) \end{pmatrix} .$$

Under the symmetry transformation, $\begin{cases} \psi \to U\psi \\ \bar{\psi} \to \bar{\psi} U^\dagger \end{cases}$

where $U = e^{i\theta^a T^a}$.

If θ^a is independent of x then the Dirac Lagrangian is invariant:

$$\bar{\psi}(i\not{\partial} - m)\psi \to \bar{\psi} U^\dagger (i\not{\partial} - m) U\psi$$
$$= \bar{\psi}(i\not{\partial} - m)\psi$$

If θ^a depends on x then the Dirac Lagrangian transforms as

$$\bar{\psi}(i\not{\partial} - m)\psi \to \bar{\psi} U^\dagger i\not{\partial}(U\psi) - m\bar{\psi}\psi$$
$$= \bar{\psi}(i\not{\partial} - m)\psi + i\bar{\psi}\gamma^\mu (U^\dagger \partial_\mu U)\psi$$

To cancel the extra term we introduce a set of gauge fields A_μ^a, $a = 1, \ldots, \dim G$.

Consider $\mathcal{L} = \overline{\Psi}(i\partial\!\!\!/ - m)\psi - e A_\mu^a \overline{\Psi}\gamma^\mu T^a \psi$.

If under a gauge transformation $A_\mu \equiv A_\mu^a T^a$ transforms as

$$\boxed{A_\mu \to U A_\mu U^+ - \frac{i}{e} U \partial_\mu U^+}$$

then

$$\begin{aligned}
\mathcal{L} \to\ & \overline{\Psi}(i\partial\!\!\!/ - m)\psi + i\overline{\Psi}\gamma^\mu (U^+\partial_\mu U)\psi \\
& - e\overline{\Psi}\gamma^\mu U^+\left(U A_\mu U^+ - \frac{i}{e} U \partial_\mu U^+\right) U \psi \\
&= \mathcal{L} + i\overline{\Psi}\gamma^\mu (U^+\partial_\mu U)\psi + i\overline{\Psi}\gamma^\mu(\partial_\mu U^+) U\psi \\
&= \mathcal{L} + i\overline{\Psi}\gamma^\mu \partial_\mu (\overbrace{U^+U}^{1})\psi \\
&= \mathcal{L}.
\end{aligned}$$

We have shown that $\mathcal{L} = \overline{\Psi}(iD\!\!\!/ - m)\psi$ is gauge invariant, where the covariant derivative is

$$\boxed{D_\mu \psi = \left(\partial_\mu + ie A_\mu^a T^a\right)\psi.}$$

Under a gauge transformation $\boxed{D_\mu \psi \to U D_\mu \psi}$, which also explains why

$$i\overline{\Psi}D\!\!\!/\psi \to i\overline{\Psi}U^+ U D\!\!\!/\psi = i\overline{\Psi}D\!\!\!/\psi.$$

To construct a gauge-covariant field strength, consider the commutator $[D_\mu, D_\nu]\psi$.

Under a gauge transformation

$$[D_\mu, D_\nu]\psi \to U[D_\mu, D_\nu]\psi .$$

Evaluating the commutator,

$$[D_\mu, D_\nu]\psi = \cancel{[\partial_\mu, \partial_\nu]}\psi + ie\left([\partial_\mu, A_\nu^a T^a] - [\partial_\nu, A_\mu^a T^a]\right)\psi - e^2 [A_\mu^a T^a, A_\nu^b T^b]\psi$$

$$= ie\underbrace{\left(\partial_\mu A_\nu^a T^a - \partial_\nu A_\mu^a T^a + ie[A_\mu^a T^a, A_\nu^b T^b]\right)}_{F_{\mu\nu}^a T^a}\psi$$

Using $[T^a, T^b] = if^{abc}T^c$, we can write the field strength as

$$\boxed{F_{\mu\nu}^a = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a - ef^{abc}A_\mu^b A_\nu^c}$$

Since under a gauge transformation

$$[D_\mu, D_\nu]\psi \to U[D_\mu, D_\nu]\psi \quad \text{and}$$

$$\psi \to U\psi ,$$

we deduce the gauge transformation of the field strength:

$$\boxed{F_{\mu\nu}^a T^a \to U F_{\mu\nu}^a T^a U^\dagger}$$

Hence, the field strength transforms homogeneously in the adjoint representation of the gauge group.

$F_{\mu\nu}^a$ is not gauge invariant, but defining $\boxed{F_{\mu\nu} = F_{\mu\nu}^a T^a},$

$$\mathrm{Tr}\, F_{\mu\nu} F^{\mu\nu} = F_{\mu\nu}^a F^{\mu\nu b} \underbrace{\mathrm{Tr}\, T^a T^b}_{C(r)\delta^{ab}}$$

$$= C(r) F_{\mu\nu}^a F^{\mu\nu a} \quad \text{is gauge invariant:}$$

$$\mathrm{Tr}\, F_{\mu\nu} F^{\mu\nu} \to \mathrm{Tr}\left(U F_{\mu\nu} U^\dagger U F^{\mu\nu} U^\dagger\right)$$

$$= \mathrm{Tr}\, F_{\mu\nu} F^{\mu\nu} \text{ by cyclicity of the trace.}$$

We arrive at a gauge-invariant Lagrangian with dynamical gauge fields:

$$\boxed{\mathcal{L}_{YM} = -\frac{1}{4} F_{\mu\nu}^a F^{\mu\nu a} + \overline{\Psi}(i\not{D} - m)\Psi}$$ Yang–Mills Lagrangian.

If T^a are generators in the fundamental representation, with $\mathrm{Tr}\, T^a T^b = \frac{1}{2}\delta^{ab}$, then we can write

$$\boxed{\mathcal{L}_{YM} = -\frac{1}{2} \mathrm{Tr}(F_{\mu\nu} F^{\mu\nu}) + \overline{\Psi}(i\not{D} - m)\Psi}$$

<u>Note</u>: To compare the above discussion w/ Peskin & Schroeder, replace the coupling $e \to -g$.

If the group G generated by T^a is $SU(3)$, and the three fields Ψ_i, $i=1,2,3$ transform in the fundamental representation of $SU(3)$, then this is Quantum Chromodynamics (QCD), and Ψ_i are the three "colors" of quark fields.

Quantization of Yang-Mills Theory

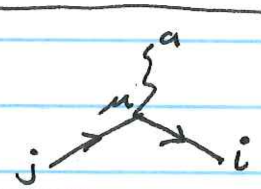
To do justice to the subject we will apply the functional integral approach to quantization.
Aside from the gauge field propagators we can anticipate most of the Feynman rules.

Fermion propagator : $\mathcal{L}_0^{\text{fermion}} = \sum_j \bar{\Psi}_j (i\not{\partial} - m)\Psi_j$

$$\langle 0 | T\, \Psi_i(x)\, \bar{\Psi}_j(y) | 0 \rangle = \int \frac{d^4k}{(2\pi)^4} \frac{i}{\not{k} - m} \delta_{ij}\, e^{-ik\cdot(x-y)}$$

$$\boxed{i \xrightarrow{k} j \quad = \frac{i}{\not{k}-m}\delta_{ij}}$$

Fermion – gauge boson vertex : $\mathcal{L} \supset -e A_\mu^a \bar{\Psi}\gamma^\mu T^a \psi$



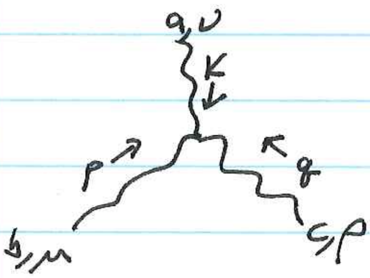
$= -ie\gamma^\mu T^a_{ij}$

The essential difference between the non-Abelian gauge theory and QED lies in the nonlinear couplings among the gauge bosons.

As a result of these nonlinear couplings QCD is asymptotically free : The effective QCD gauge coupling is small at high energies. In QED the effective coupling is small at low energies.

Triple-gauge boson vertex: $\mathcal{L} \supset e f^{abc} (\partial_\mu A_\nu^a) A^{\mu b} A^{\nu c}$

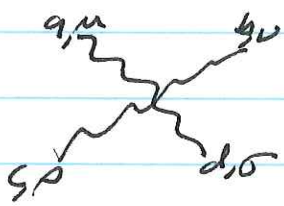
There are $3! = 6$ ways to contract the gauge fields $A_\mu^a, A_\mu^b, A_\mu^c$ with the other gauge bosons attached to the ends of the lines attached to the vertex. If we label momenta as ingoing, then $\partial_\mu \to -ik_\mu$ when acting on a line w/ momentum k_μ. There are 6 terms, with signs alternating due to the antisymmetry of f^{abc}.



$$= -e f^{abc}\left[g^{\mu\rho}(p-q)^\nu + g^{\nu\rho}(q-k)^\mu + g^{\mu\nu}(k-p)^\rho\right]$$

Quadruple-gauge boson vertex: $\mathcal{L} \supset -e^2 f^{abc} f^{ade} A_\mu^b A_\nu^c A^{\mu d} A^{\nu e}$

There are $4!$ contractions which we can collect in 6 terms:



$$= -ie^2\left[f^{abe}f^{cde}(g^{\mu\rho}g^{\nu\sigma} - g^{\mu\sigma}g^{\nu\rho}) + f^{ace}f^{bde}(g^{\mu\nu}g^{\rho\sigma} - g^{\mu\sigma}g^{\nu\rho}) + f^{ade}f^{bce}(g^{\mu\nu}g^{\rho\sigma} - g^{\mu\rho}g^{\nu\sigma})\right]$$