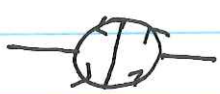
General Loop Diagrams

We already have the tools to express any one-loop Feynman diagram as an integral over a finite number of Feynman parameters. We can generalize those techniques to multi-loop diagrams.

Consider a diagram w/ L loops and I internal lines,

eg. 

$L = 2 \rightarrow \int d^4k_1\, d^4k_2$

$I = 5 \rightarrow 5$ propagators

The denominators from the propagators can be combined by a Feynman parametrization. Then the diagram is of the form,

$$\int_0^1 dx_1 \cdots dx_I\, \delta(1 - \Sigma x_i) \int \frac{d^4k_1 \cdots d^4k_L}{(2\pi)^{4L}\, D^I} N .$$

where D is of the form,

$$D = \sum_{i,j=1}^{L} A_{ij}\, k_i \cdot k_j + \sum_{i=1}^{L} B_i \cdot k_i + C$$

A_{ij}: $L \times L$ matrix fn. of Feynman parameters

B_i: vector w/ L 4-vector components

C: #, linearly dependent on Feynman parameters, external momenta^2, masses^2. $\mathrm{Im}(C) = i\epsilon$

N = function of momenta, γ-matrices.

Next, complete the squares: Eliminate the terms linear in k_i by shifting the integrations over k_i:

$$k_i' = k_i + \frac{1}{2}\sum_j (A^{-1})_{ij} B_j \quad , \quad d^4k_i = d^4k_i'$$

$$D = \sum_{ij=1}^{L} A_{ij}\, k_i' \cdot k_j' + C' \quad , \quad C' = C - \frac{1}{4}\sum_{ij} B_i A_{ij}^{-1} B_j$$

Next diagonalize A_{ij} w/ an orthogonal transformation on the 4-vectors k_i':

$$k_i' = O_{ij}\, k_j'' \quad , \quad \det O = 1$$

$$\prod_{i=1}^{L} d^4k_i' = \prod_{i=1}^{L} d^4k_i''$$

$$\left.\begin{aligned} D &= \sum_{\substack{ij=1\\kl}}^{L} A_{ij}\, O_{ik}\, k_k'' \cdot k_l''\, O_{jl} + C' \\ &= \sum_{kl=1}^{L} (O^T A O)_{kl}\, k_k'' \cdot k_l'' + C' \\ &= \sum_{i=1}^{L} a_i\, k_i'' \cdot k_i'' + C' \end{aligned}\right\}$$

Note: In the end it won't matter what O_{ij} or a_i are. Only ($\det A$) will appear in the final result.

where O is chosen s.t. $(O^T A O)_{kl} = a_l \delta_{kl}$

Finally, redefine k_i'' to eliminate the a_i:

$$k_i'' = \frac{1}{\sqrt{a_i}} k_i'''$$

$$\prod_{i=1}^{L} d^4 k_i''' = \prod_{i=1}^{L} (\sqrt{a_i})^4 d^4 k_i''$$

$$= (\det A)^2 \prod_{i=1}^{L} d^4 k_i''$$

$$D = \sum_{i=1}^{L} k_i''' \cdot k_i''' + C'$$

The integral to be done is now of the form

$$\int_0^1 dx_1 \cdots dx_I \, \delta(1-\sum x_i) \int (\det A)^{-2} \quad \frac{d^4 k_1''' \cdots d^4 k_L'''}{(2\pi)^{4L} D^I} N(k_i''')$$

The terms in N that are odd in k_i''' integrate to 0. Symmetry arguments can often simplify the terms in N even in k_i'''.

To make the integrals spherically symmetric (rather than Lorentz symmetric), Wick rotate each of the $k_i^{0\,'''}$. The integral becomes

$$\int_0^1 dx_1 \cdots dx_I \, \delta(1-\sum x_i) \int \frac{d^4 k_{1E} \cdots d^4 k_{LE}}{(2\pi)^{4L} \; D^I} N(k_{iE}) (\det A)^{-2} \, i^L$$

$$d^4 k_i''' = i \, d^4 k_{iE} \, , \quad k_{i4} = -i k_i^{0\,'''}$$

$$D = -\sum_{i=1}^{L} k_{iE}^2 + C'$$

Sphericial symmetry allows you to perform the 4L momentum integrals (if you are lucky and clever).

The result is an integral over I Feynman parameters, or I-1 parameters if you do the trivial δ-function integral over one of the I parameters.