Electron Vertex Function: Part 2

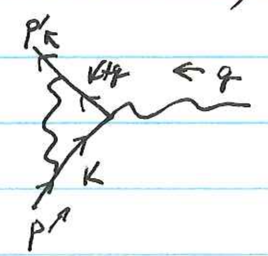
We have seen that the electon vertex function in QED w/ on-shell electrons can be decomposed into two terms:

$-i\tilde{\Gamma}^\mu(p,p') =$ 

$$= -ie\gamma^\mu F_1(q^2) + \frac{\sigma^{\mu\nu}}{2m} q_\nu e F_2(q^2)$$

Since e is the physical electron charge, the counterterm $\mathcal{L}_{CT} \supset -eD\bar{\Psi}\not{A}\Psi$ should be chosen such that $\boxed{F_1(0)=1}$. As mentioned earlier, gauge invariance relates this counterterm to the electron wavefunction renormalization counterterm, $\mathcal{L}_{CT} \supset D\bar{\Psi}i\not{\partial}\Psi$. As we will see, the same counterterm that sets $\left.\frac{d\Sigma}{d\not{p}}\right|_{\not{p}=m} = 0$ also sets $F_1(0)=1$.

The Pauli form factor $F_2(q^2)$ contains information about the electron's magnetic moment. At lowest order in e, $F_2(q^2)$ comes from the diagram



$\equiv \delta(-i\Gamma^\mu)$

$\bar{u}(p')\,\delta(-i\Gamma^\mu)\,u(p)$

$$= \int \frac{d^4k}{(2\pi)^4} \left[\frac{(-ig_{\nu\lambda})}{(p-k)^2+i\epsilon}\, \bar{u}(p')(-ie\gamma^\nu) \frac{i(\not{k}+\not{q}+m)}{(k+q)^2-m^2+i\epsilon} (-ie\gamma^\mu) \right.$$

$$\left. \times \frac{i(\not{k}+m)}{k^2-m^2+i\epsilon} (-ie\gamma^\lambda)\, u(p) \right]$$

Using the γ-matrix contraction identities this can be simplified: (<u>Exercise</u>)

$\bar{u}(p')\,\delta(-i\Gamma^\mu)\,u(p)$

$$= 2e^2 \int \frac{d^4k}{(2\pi)^4} \frac{\bar{u}(p')\left[\not{k}\gamma^\mu(\not{k}+\not{q}) + m^2\gamma^\mu - 2m\,(2k+q)^\mu\right]u(p)}{\left((p-k)^2+i\epsilon\right)\left((k+q)^2-m^2+i\epsilon\right)\left(k^2-m^2+i\epsilon\right)}$$

We could proceed as in our previous self-energy calculations, except that we can't combine the three factors in the denominator with a single Feynman parameter.

We want to generalize $\frac{1}{A_1+i\epsilon}\frac{1}{A_2+i\epsilon} = \int_0^1 dx\left[A_1 x + A_2(1-x)+i\epsilon\right]^{-2}$

The trick uses the function $\Gamma(x)$:

$$\Gamma(x) = \int_0^\infty dt\; t^{x-1} e^{-t}$$

$$\int_0^\infty dt\; t^{\alpha-1} e^{-At} = \frac{1}{A^\alpha}\Gamma(\alpha)$$

$$\frac{1}{\prod_j (A_j^{\alpha_j})} = \prod_j \frac{1}{\Gamma(\alpha_j)} \int_0^\infty dt_j\; t_j^{\alpha_j-1} e^{-A_j t_j} \underbrace{\int_0^\infty ds\, \delta(s-\sum_i t_i)}_{1}$$

Let $t_j = s x_j$:

$$\frac{1}{\prod_j (A_j^{\alpha_j})} = \int_0^\infty ds \prod_j \frac{1}{\Gamma(\alpha_j)} \int_0^\infty dx_j\, s^{\alpha_j} x_j^{\alpha_j-1} e^{-sA_j x_j} \frac{\delta(1-\sum x_j)}{s}$$

$$= \prod_j \left(\frac{1}{\Gamma(\alpha_j)}\right) \int_0^1 \left(\prod_j dx_j \prod_j x_j^{\alpha_j-1}\right) \int_0^\infty ds\; s^{(\sum_i \alpha_i - 1)} e^{-s\sum_i x_i A_i} \delta(1-\sum_i x_i)$$

$$= \left(\prod_j \frac{1}{\Gamma(\alpha_j)}\right) \int_0^1 \prod_j dx_j \left(\prod_j x_j^{\alpha_j-1}\right) \frac{\Gamma(\sum_i \alpha_i)}{\left(\sum_i x_i A_i\right)^{\sum \alpha_i}} \delta(1-\sum_i x_i)$$

$$\boxed{\frac{1}{\prod_j A_j^{\alpha_j}} = \frac{\Gamma(\sum \alpha_i)}{\prod_j \Gamma(\alpha_j)} \int_0^1 dx_1 \cdots dx_n\, \delta(1-\sum_i x_i) \frac{\prod_j x^{\alpha_j-1}}{\left(\sum_i x_i A_i\right)^{\sum \alpha_i}}}$$

<u>Example</u>: $\dfrac{1}{A+i\epsilon}\dfrac{1}{B+i\epsilon}$ $\qquad A_1 = A+i\epsilon\;,\; \alpha_1 = 1$

$\qquad\qquad\qquad\qquad\qquad\qquad A_2 = B+i\epsilon\;,\; \alpha_2 = 1$

$$= \frac{\Gamma(2)}{\Gamma(1)\Gamma(1)} \int dx_1\, dx_2\, \delta(1-x_1-x_2) \frac{1}{[(A+i\epsilon)x_1 + (B+i\epsilon)x_2]^2}$$

$$= \int_0^1 dx\, [(A+i\epsilon)x + (B+i\epsilon)(1-x)]^{-2}$$

$$= \int_0^1 dx\, [Ax + B(1-x) + i\epsilon]^{-2}$$ This is the formula we used before.

The general formula for Feynman parametrization of denominators in loop integrals is:

$$\boxed{\prod_{j=1}^{n} \frac{1}{(a_j + i\epsilon)} = (n-1)! \int_0^1 dx_1 \int_0^{1-x_1} dx_2 \int_0^{1-x_1-x_2} dx_3 \cdots \int_0^{1-x_1-\cdots-x_{n-2}} dx_{n-1} \cdot \left[\sum_{j=1}^{n-1} x_j a_j + \left(1 - \sum_{j=1}^{n-1} x_j\right) a_n + i\epsilon \right]^{-n}}$$

Using this, we can: (1) Combine the denominators in the 1-loop vertex function.

(2) Complete the square to make use of the Lorentz symmetry of the integrand, as usual.

(3) Manipulate the numerator to a sum of one term $\propto \gamma^\mu$ and one term $\propto \sigma^{\mu\nu} q_\nu$.

(4) Regularize divergent integrals using an acceptable regularization procedure (consistent with gauge invariance).

You will do all of this for homework.

We will first discuss the result for the Pauli form factor $F_2(q^2)$, which is:

$$\boxed{F_2(q^2) = \frac{e^2}{8\pi^2} \int_0^1 dx\, dy\, dz\, \delta(x+y+z-1) \frac{2m^2 z(1-z)}{-q^2 xy + m^2(1-z)^2}}$$

From $F_2(q^2)$ we can calculate the anomalous magnetic moment of the electron, to $O(e^2)$:

$$F_2(0) = \frac{g-2}{2} = \frac{e^2}{8\pi^2}\int_0^1 dx\,dy\,dz\,\delta(x+y+z-1)\,\frac{2z}{1-z}$$

$$= \frac{e^2}{8\pi^2}\int_0^1 dz\int_0^{1-z} dx\,\frac{2z}{1-z}$$

$$= \frac{e^2}{8\pi^2}\int_0^1 2z$$

$$= \frac{e^2}{8\pi^2} = \frac{\alpha}{2\pi}$$

With $\alpha = \frac{1}{137.036}$, we obtain

$$\boxed{\frac{g-2}{2} \approx \frac{\alpha}{2\pi}} \approx 0.001161$$

The digit that changes from higher order corrections.

The magnitude of the electron magnetic moment is then,

$$\mu = \frac{e}{2m}\left[1 + \frac{\alpha}{2\pi} + O\left(\frac{\alpha}{\pi}\right)^2\right]$$

$$= \frac{e}{2m}(1.001161)$$

This is a theoretical calculation to 6 digits.

Experiment gives $\mu = \frac{e}{2m}(1.00115965218077 \pm 0.00000000000027)$

It seems that QED works!