Dimensional Regularization with γ-matrices

For Lorentz invariance we require the Clifford algebra

$$\boxed{\begin{aligned}\{\gamma^\mu, \gamma^\nu\} &= 2g^{\mu\nu} \\ \mathrm{Tr}\,\mathbb{1} &= 4\end{aligned}}$$

— or $2^{d/2}$ in even dims, $2^{\frac{d-1}{2}}$ in odd dims (see homework)

The metric satisfies $\boxed{g_{\mu\nu}g^{\mu\nu} = d}$

From the Clifford algebra we deduce the γ-matrix contraction identities in d dimensions:

$$\gamma^\mu\gamma_\mu = d$$
$$\gamma^\mu\gamma^\nu\gamma_\mu = -(d-2)\gamma^\nu$$
$$\gamma^\mu\gamma^\nu\gamma^\rho\gamma_\mu = 4g^{\nu\rho} - (4-d)\gamma^\nu\gamma^\rho$$
$$\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma\gamma_\mu = -2\gamma^\sigma\gamma^\rho\gamma^\nu + (4-d)\gamma^\nu\gamma^\rho\gamma^\sigma$$

Again, it is important not to throw out the terms that vanish as $d \to 4$ until the end of the calculation, as they may multiply poles and become nonzero and finite.

There are different ways to handle γ^5. 't Hooft and Veltman set $\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3$, just like in $d=4$. Then $\{\gamma^5, \gamma^\mu\} = 0$ for $\mu = 0,1,2,3$, but not for $\mu > 3$.
Alternatively, we can treat γ^5 formally as satisfying $\{\gamma^5, \gamma^\mu\} = 0 \ \forall \mu$.

Now back to the photon self energy.
We have derived the following expression at one-loop:

$\equiv i\Pi^{\mu\nu}(q)$

$$= -e^2 \cdot 4 \int_0^1 dx \int \frac{d^4 l}{(2\pi)^4} \left[2l^\mu l^\nu - g^{\mu\nu} l^2 + 2x^2 q^\mu q^\nu - 2x q^\mu q^\nu + g^{\mu\nu}(m^2 + x q^2 - x^2 q^2) \right] \left[l^2 + x(1-x) q^2 - m^2 + i\epsilon \right]^{-2}$$

In d-dimensions we can replace $l^\mu l^\nu \to \frac{1}{d} l^2 g^{\mu\nu}$.
(The $g^{\mu\nu}$ is from Lorentz invariance and symmetry in $\mu \leftrightarrow \nu$.
The $\frac{1}{d}$ is from taking the trace of both sides.)

Then, (can set $d=4$ here because it comes in a combination that doesn't vanish as $d \to 4$.)

$$i\Pi^{\mu\nu}(q) = -4e^2 \int_0^1 dx \int \frac{d^4 l}{(2\pi)^4} \left[\left(\frac{2}{4} - 1\right) l^2 g^{\mu\nu} + 2x(x-1) q^\mu q^\nu + g^{\mu\nu}(m^2 + q^2 x(1-x)) \right] \left[l^2 + x(1-x) q^2 - m^2 + i\epsilon \right]^{-2}$$

Wick rotate: $d^4 l \to i\, d^4 l_E$, $\quad l^2 \to -l_E^2$

$$i\pi^{\mu\nu}(q) = -4ie^2 \int_0^1 dx \int \frac{d^4 l_E}{(2\pi)^4} \frac{\left[\frac{1}{2} g^{\mu\nu} l_E^2 - 2x(1-x) q^\mu q^\nu + g^{\mu\nu}(m^2 + x(1-x) q^2) \right]}{\left[-l_E^2 + x(1-x) q^2 - m^2 + i\epsilon \right]^2}$$

We use dimensional regularization to regulate the divergent integral.

In d dimensions the electric charge e has mass dimension $[e] = \frac{4-d}{2}$, so we replace e by $e\mu^{\frac{4-d}{2}}$ for some arbitrary mass scale μ.

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Check: $\left[\int d^dx\, \bar{\psi} i\partial\!\!\!/ \psi\right] = -d + 2[\psi] + 1 = 0$

$$\rightarrow [\psi] = \frac{d-1}{2}$$

$$\left[\int d^dx\, F_{\mu\nu}F^{\mu\nu}\right] = -d + 2[A^\mu] + 2 = 0$$

$$\rightarrow [A^\mu] = \frac{d-2}{2}$$

$$\left[\int d^dx\, e\bar{\psi}A\!\!\!/\psi\right] = [e] + 2\cdot\frac{d-1}{2} + \frac{d-2}{2} - d = 0$$

$$\rightarrow [e] = \frac{4-d}{2}$$

We use our dim reg integral table:

$$\boxed{\begin{aligned} \int \frac{d^d l_E}{(l_E^2+a^2)^n} &= \frac{\pi^{d/2}}{\Gamma(n)} \frac{\Gamma(n-d/2)}{a^{2n-d}} \\ \int \frac{d^d l_E\, l_E^2}{(l_E^2+a^2)^n} &= \frac{\pi^{d/2}}{\Gamma(n)} \frac{d}{2} \cdot \frac{\Gamma(n-\frac{d}{2}-1)}{a^{2n-d-2}} \end{aligned}}$$

We already showed the first of these integrals. The second is left as an exercise.

It is conventional (but not necessary) to replace the $\frac{1}{(2\pi)^4}$ in the integral by $\frac{1}{(2\pi)^d}$.

We now have,

$$i\Pi^{\mu\nu}(q) = -4ie^2\mu^{4-d}\int_0^1 dx\left[\overset{(1-\frac{2}{d})}{\frac{1}{2}}g^{\mu\nu}\cdot\frac{1}{(4\pi)^{d/2}}\frac{\Gamma(2-d/2-1)}{\Gamma(2)}\cdot\frac{d}{2}\right.$$

$$(m^2-x(1-x)q^2)^{\frac{2+d-4}{2}}$$

$$+\left[g^{\mu\nu}(m^2+x(1-x)q^2)-2x(1-x)q^\mu q^\nu\right]\frac{1}{(4\pi)^{d/2}}\frac{\Gamma(2-d/2)}{\Gamma(2)}$$

$$\left.\cdot(m^2-x(1-x)q^2)^{\frac{d-4}{2}}\right]$$

$$= -4ie^2\mu^{4-d}\int_0^1 dx\left[\frac{1}{(4\pi)^{d/2}}\cdot\Gamma(2-d/2)\,(m^2-x(1-x)q^2)\,g^{\mu\nu}\right.$$

$$\left.+\frac{1}{(4\pi)^{d/2}}\Gamma(2-d/2)\left(g^{\mu\nu}(m^2+x(1-x)q^2)-2x(1-x)q^\mu q^\nu\right)\right]$$

$$\times\,(m^2-x(1-x)q^2)^{\frac{d-4}{2}}$$

$$= -4ie^2\int_0^1 dx\,\frac{1}{(4\pi)^{d/2}}\Gamma(2-d/2)\;2x(1-x)\left(\frac{\mu^2}{m^2-x(1-x)q^2}\right)^{\frac{4-d}{2}}$$

$$\times\,(g^{\mu\nu}q^2-q^\mu q^\nu)$$

Note that our result is transverse. Dim reg is a gauge invariant regulator, and preserves consequences of gauge invariance.

We define $i\Pi^{\mu\nu}(q) = i(g^{\mu\nu}q^2-q^\mu q^\nu)\,\Pi(q^2)$

$$\boxed{\Pi(q^2) = -\frac{8e^2}{(4\pi)^{d/2}}\Gamma(2-d/2)\int_0^1 dx\, x(1-x)\left(\frac{\mu^2}{m^2-x(1-x)q^2}\right)^{\frac{4-d}{2}}}$$

To pick out the divergence we expand $\Gamma(2-d/2)$:

$$\Gamma(2-d/2) = \frac{2}{4-d} - \gamma_E + \mathcal{O}(4-d)$$

$$\left(\frac{\mu^2}{m^2-x(1-x)q^2}\right)^{\frac{4-d}{2}} = 1 + \frac{4-d}{2}\log\left(\frac{\mu^2}{m^2-x(1-x)q^2}\right) + \mathcal{O}(4-d)^2$$

$$\Gamma(2-d/2)\left(\frac{\mu^2}{m^2-x(1-x)q^2}\right)^{\frac{4-d}{2}} = \frac{2}{4-d} - \gamma_E + \log\left(\frac{\mu^2}{m^2-x(1-x)q^2}\right) + \mathcal{O}(4-d)$$

Then,

$$\boxed{\Pi(q^2) = \frac{-8e^2}{(4\pi)^2}\int_0^1 dx\, x(1-x)\left(\frac{2}{4-d} - \gamma_E - \log\left(\frac{m^2-x(1-x)q^2}{\mu^2}\right)\right) + \mathcal{O}(4-d)}$$

The counterterm $\mathcal{L}_{CT} \supset -\frac{A}{4}F_{\mu\nu}F^{\mu\nu}$ shifts $\Pi(q^2)$ by a constant. This allows us to satisfy our renormalization condition $\tilde{\Pi}(0)=0$:

$$\boxed{\tilde{\Pi}(q^2) = \Pi(q^2) - \Pi(0) = +\frac{e^2}{2\pi^2}\int_0^1 dx\, x(1-x)\log\left(\frac{m^2-x(1-x)q^2}{m^2}\right)}$$

Physical Interpretation of $\tilde{\Pi}(q^2)$

The expression we have derived for $\tilde{\Pi}(q^2)$ carries important physical information.

First of all, consider the relation between the physical coupling e^2 and the bare coupling e_0^2:

$$e^2 = Z_3 e_0^2 .$$

The counterterm contributes to $Z_3 = 1 + A$, with $A = \Pi(0)$, from  $= -iAq^2\left(g^{\mu\nu} - \frac{q^\mu q^\nu}{q^2}\right)$.

In our case, $\Pi(0) = \frac{e^2}{2\pi^2}\int_0^1 dx\, x(1-x)\left[\log\left(\frac{m^2}{\mu^2}\right) + \gamma_E - \frac{2}{4-d}\right]$

The dominant term is the divergent term, which is negative as $d \to 4$ from below. Hence, to this order $\boxed{Z_3 < 1.}$

It follows that $\boxed{e^2 < e_0^2.}$ We can think of the difference between the bare and physical charge as due to <u>screening</u> of charge by the polarization of the vacuum. If you put a charged particle in the vacuum, it acts as a source of the electromagnetic field. The vacuum can also contain e^+e^- pairs, which become polarized in the presence of the bare charge of the particle. This is why screening happens, and if you measure the physical charge of a particle from far away, it will be < bare charge.

Vacuum Polarization

The dependence of the effective charge on q^2, $\alpha_{eff}(q^2) = \dfrac{\alpha}{1-\tilde{\Pi}(q^2)}$, can be translated into a distance-dependent correction to the Coulomb potential.

Consider a static source with charge e. Recall the Coulomb potential,

$$V(\vec{r}) = \int \frac{d^3q}{(2\pi)^3} \frac{-e^2}{\vec{q}^{\,2}} e^{i\vec{q}\cdot\vec{r}}$$

$$= -\frac{\alpha}{r} \quad , \alpha = \frac{e^2}{4\pi} .$$

We now have,

$$V(\vec{r}) = \int \frac{d^3q}{(2\pi)^3} \frac{-e^2 \, e^{i\vec{q}\cdot\vec{r}}}{\vec{q}^{\,2}\left[1-\tilde{\Pi}(-\vec{q}^{\,2})\right]}$$

For $q^2 \ll m^2$ we can expand $\tilde{\Pi}(q^2)$:

$$\tilde{\Pi}(q^2) = \frac{2\alpha}{\pi}\int_0^1 dx \; x(1-x)\log\left(1-\frac{x(1-x)q^2}{m^2}\right)$$

$$\approx -\frac{2\alpha}{\pi}\int_0^1 dx \; x^2(1-x)^2\frac{q^2}{m^2} \quad = -\frac{\alpha}{15\pi}\frac{q^2}{m^2}$$

Then, $$V(\vec{r}) \approx \int \frac{d^3q}{(2\pi)^3} e^{i\vec{q}\cdot\vec{r}} (-4\pi\alpha) \left[\frac{1}{\vec{q}^2} + \frac{\alpha}{15\pi m^2} \right]$$

$$= -\frac{\alpha}{r} - \frac{4\alpha^2}{15m^2} \delta^3(\vec{r})$$

The correction to the Coulomb potential is evidently small at large distances, but important at shorter distances. This effect is observable, though, for example in the spectrum of the Hydrogen atom. The energy levels get a shift,

$$\Delta E \approx \int d^3x \; |\psi(\vec{x})|^2 \left(-\frac{4\alpha^2}{15m^2} \delta^3(\vec{x}) \right)$$

$$= -\frac{4\alpha^2}{15m^2} |\psi(0)|^2$$

Only S-wave states have nonvanishing $\psi(0)$. Hence, there is a splitting between the $2S_{1/2}$ and $2P_{1/2}$ states. This is part of the <u>Lamb shift</u>.

We can do better than this δ-function approximation to the corrections to the Coulomb potential.

$$V(\vec{r}) \approx \int \frac{d^3q}{(2\pi)^3} \; \frac{-e^2 e^{i\vec{q}\cdot\vec{r}}}{\vec{q}^2} \left(1 + \hat{\Pi}(-\vec{q}^2) \right)$$

$$= \frac{ie^2}{(2\pi)^2} \frac{1}{r} \int_{-\infty}^{\infty} dQ \; \frac{Qe^{iQr}}{Q^2} \left(1 + \hat{\Pi}(-Q^2) \right)$$

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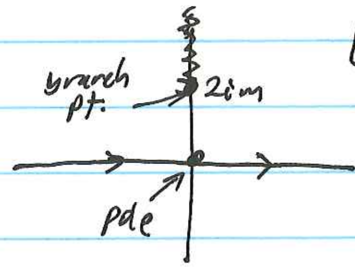
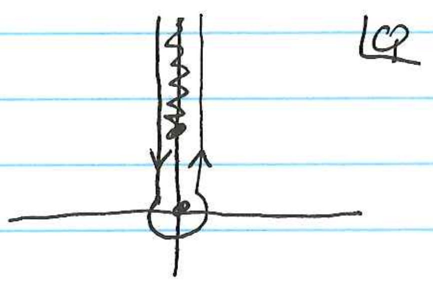
The integrand has a pole at $Q=0$ and a branch cut starting at $Q=2im$ from the log in $\hat{\pi}(-Q^2)$.



Stretch the contour to go along the imaginary axis:



The integral around the pole gives the Coulomb potential. The real part of the integrand is the same on both sides of the cut, but the imaginary part is not.

Consider $\int_0^1 \log\left(\frac{m^2 - x(1-x)q^2}{m^2}\right) x(1-x)\, dx$.

Choose a branch of the log which is real when its argument is positive, with the branch cut along the imaginary axis, beginning at $q^2 = 4m^2$, $\mathrm{Im}\left[\log(-X \pm i\epsilon)\right] = \pm\pi$. The values of x for which the arg of the log is negative is $\frac{1}{2} - \frac{1}{2}\beta < x < \frac{1}{2} + \frac{1}{2}\beta$, where $\beta = \sqrt{1 - 4m^2/q^2}$. Then

$$\mathrm{Im}\left[\hat{\pi}(q^2 \pm i\epsilon)\right] = \frac{-2\alpha}{\pi}(\pm\pi)\int_{\frac{1}{2}-\frac{1}{2}\beta}^{\frac{1}{2}+\frac{1}{2}\beta} dx\, x(1-x) = \mp\frac{\alpha}{3}\sqrt{1-\frac{4m^2}{q^2}}\left(1+\frac{2m^2}{q^2}\right).$$

The contribution to $V(\vec{r})$ from the integration up and down the branch cut is,

$$\delta V(\vec{r}) = \frac{-e^2}{4\pi^2 r} \cdot 2\int_{2m}^{\infty} dq\, \frac{e^{-qr}}{q}\, \mathrm{Im}\left[\tilde{\Pi}(q^2 - i\epsilon)\right]$$

where we have changed variables $Q = iq$.

$$\delta V(\vec{r}) = \frac{-e^2}{2\pi^2 r} \int_{2m}^{\infty} dq\, \frac{e^{-qr}}{q}\, \frac{\alpha}{3}\sqrt{1 - \frac{4m^2}{q^2}}\left(1 + \frac{2m^2}{q^2}\right)$$

For $r \gg 1/m$, the integral is dominated by the region around $q \approx 2m$. Expanding the square root about $q = 2m$ and changing variables $t = q - 2m$,

$$\delta V(\vec{r}) = -\frac{\alpha}{r} \cdot \frac{2}{\pi} \int_0^{\infty} dt \left[\frac{e^{-(t+2m)r}}{2m}\, \frac{\alpha}{3}\sqrt{\frac{t}{m}} \cdot \frac{3}{2} + \mathcal{O}(t)\right]$$

$$\approx -\frac{\alpha}{r}\, \frac{\alpha}{4\sqrt{\pi}}\, \frac{e^{-2mr}}{(mr)^{3/2}} \qquad \underline{\text{Uehling potential}}$$

Adding the Coulomb term from the integral around the pole,

$$\boxed{V(r) \approx -\frac{\alpha}{r}\left(1 + \frac{\alpha}{4\sqrt{\pi}}\, \frac{e^{-2mr}}{(mr)^{3/2}} + \cdots\right)}$$

At shorter distances $r \ll \frac{1}{m}$ the region $-q^2 \gg m^2$ is most important. Then,

$$\tilde{\pi}(q^2) \simeq \frac{2\alpha}{\pi}\int_0^1 dx\, x(1-x)\left[\log\left(\frac{-q^2}{m^2}\right) + \log\left(x(1-x)\right) + \mathcal{O}\left(\frac{m^2}{q^2}\right)\right]$$

$$= \frac{\alpha}{3\pi}\left[\log\left(\frac{-q^2}{m^2}\right) - \frac{5}{3} + \mathcal{O}\left(\frac{m^2}{q^2}\right)\right]$$

Hence, for large q^2,

$$\boxed{\alpha_{eff}(q^2) \approx \frac{\alpha}{1 - \frac{\alpha}{3\pi}\log\left(\frac{-q^2}{e^{5/3}m^2}\right)}}$$

The effective electromagnetic coupling gets larger at shorter distances, as expected from screening due to vacuum polarization. At exponentially large $-q^2/m^2$, the effective coupling will apparently become infinitely strong. This signifies a breakdown of QED, known as the <u>Landau pole</u>.