Charge Renormalization and Photon Self-Energy

Consider the renormalized QED Lagrangian w/ two charged fermions, one with charge q_1, and one w/ charge q_2:

$$\mathcal{L} = -\frac{1}{4}\tilde{F}_{\mu\nu}\tilde{F}^{\mu\nu} + \bar{\tilde{\psi}}_1(i\partial\!\!\!/ + q_1\tilde{A\!\!\!/} - m_1)\tilde{\psi}_1 + \bar{\tilde{\psi}}_2(i\partial\!\!\!/ + q_2\tilde{A\!\!\!/} - m_2)\tilde{\psi}_2$$

$$+\mathcal{L}_{CT}$$

$$\mathcal{L}_{CT} = -\frac{A}{4}\tilde{F}_{\mu\nu}\tilde{F}^{\mu\nu} + B\bar{\tilde{\psi}}_1(i\partial\!\!\!/ + q_1\tilde{A\!\!\!/})\tilde{\psi}_1 - C\bar{\tilde{\psi}}_1\tilde{\psi}_1$$

$$+ D\bar{\tilde{\psi}}_2(i\partial\!\!\!/ + q_2\tilde{A\!\!\!/})\tilde{\psi}_2 - E\bar{\tilde{\psi}}_2\tilde{\psi}_2$$

It will turn out that only gauge invariant counterterms are necessary to renormalize QED. That is why we did not include separate counterterms for $\bar{\tilde{\psi}}_1 i\partial\!\!\!/\tilde{\psi}_1$ and $\bar{\tilde{\psi}}_1\tilde{A\!\!\!/}\tilde{\psi}_1$.

Gauge symmetry: $\tilde{A}_\mu \to \tilde{A}_\mu + \partial_\mu\theta(x)$
$\tilde{\psi}_1 \to \tilde{\psi}_1\exp[iq_1\theta(x)]$
$\tilde{\psi}_2 \to \tilde{\psi}_2\exp[iq_2\theta(x)]$

Define $Z_3 = 1 + A$, $Z_2^{(1)} = 1 + B$, $Z_2^{(2)} = 1 + D$

$$\psi_1 = Z_2^{(1)\,1/2}\tilde{\psi}_1, \quad \psi_2 = Z_2^{(2)\,1/2}\tilde{\psi}_2, \quad A_\mu = Z_3^{1/2}\tilde{A}_\mu$$
$$q_0^{(1)} = Z_3^{-1/2}q_1, \quad q_0^{(2)} = Z_3^{-1/2}q_2, \quad m_0^{(1)} = Z_2^{-1}(m_1 + C), \quad m_0^{(2)} = Z_2^{-1}(m_2 + E)$$

In terms of these renormalization factors we can rewrite the Lagrangian:

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \bar{\psi}_1 \left(i\not{\partial} + q_0^{(1)} \not{A} - m_0^{(1)} \right) \psi_1 + \bar{\psi}_2 \left(i\not{\partial} + q_0^{(2)} - m_0^{(2)} \right) \psi_2$$

$q_0^{(1)}$ and $q_0^{(2)}$ are the <u>bare charges</u> of the fields ψ_1 and ψ_2.

Note that the bare charges are related to the physical renormalized charges by a factor of $Z_3^{1/2}$:

$$q_1^{(\text{physical})} = Z_3^{1/2}\, q_0^{(1)}$$

$$q_2^{(\text{physical})} = Z_3^{1/2}\, q_0^{(2)}$$

Hence, $$\boxed{\frac{q_1^{(\text{physical})}}{q_2^{(\text{physical})}} = \frac{q_0^{(1)}}{q_0^{(2)}}}$$

If an electron and an antiproton have the same bare charges, then they have the same renormalized charges. It doesn't matter that the proton participates in complicated strong interactions that look very different than the electron's interactions. Gauge invariance implies that ratios of charges are invariant under renormalization. (Note that we still need to prove the assumption i.e. only gauge invariant counterterms.)

The photon field strength renormalization, Z_3, is determined from the photon self energy $\equiv$ vacuum polarization.


$= i\,\tilde{\Pi}^{\mu\nu}(q)$ (w/ external photon propagators removed)

$\tilde{\Pi}^{\mu\nu}(q)$ contains terms with tensor structure $g^{\mu\nu}$ and $q^\mu q^\nu$. Only the term $\propto g^{\mu\nu}$ is important for S-matrix elements. Heuristically, the reason is that the photon couples to a conserved current, so the photon self energy will always be attached to a J_μ such that $q^\mu J_\mu = 0$.

The more precise statement is that, as we will discuss later, gauge invariance implies a <u>Ward identity</u> w/ the consequence $q_\mu \tilde{\Pi}^{\mu\nu}(q) = 0$.

This implies $\boxed{\tilde{\Pi}^{\mu\nu}(q) = (q^2 g^{\mu\nu} - q^\mu q^\nu)\,\tilde{\Pi}(q^2)}$

for some $\tilde{\Pi}(q^2)$.

To get the renormalized photon propagator we sum over chains of 1PI diagrams:

$$= \frac{-i g_{\mu\nu}}{q^2} + \frac{(-i g_{\mu\rho})}{q^2}\left[i(q^2 g^{\rho\sigma} - q^\rho q^\sigma)\,\tilde{\Pi}(q^2)\right]\frac{(-i g_{\sigma\nu})}{q^2} + \dots$$

Note that the factor $\left(g^{\rho\sigma} - \frac{q^\rho q^\sigma}{q^2}\right) g_{\sigma\nu}$ is a projector:

$$\left(\delta_\nu^\rho - \frac{q^\rho q_\nu}{q^2}\right)\left(\delta_\alpha^\nu - \frac{q^\nu q_\alpha}{q^2}\right) = \left(\delta_\alpha^\rho - \frac{q^\rho q_\alpha}{q^2}\right)$$

— It projects onto transverse quantities, such that $q_\mu \tilde{\Pi}^{\mu\nu} = 0$.
Using this,


$$= \frac{-i g_{\mu\nu}}{q^2} + \frac{(-i g_{\mu\rho})}{q^2}\left(\delta_\nu^\rho - \frac{q^\rho q_\nu}{q^2}\right)\tilde{\Pi}(q^2)$$
$$+ \frac{(-i g_{\mu\rho})}{q^2}\left(\delta_\nu^\rho - \frac{q^\rho q_\nu}{q^2}\right)\tilde{\Pi}^2(q^2)$$
$$+ \quad \ldots$$


$$\boxed{= \frac{-i}{q^2(1-\tilde{\Pi}(q^2))}\left(g_{\mu\nu} - \frac{q_\mu q_\nu}{q^2}\right) + \frac{-i}{q^2}\left(\frac{q_\mu q_\nu}{q^2}\right)}$$

The renormalized propagator has developed a longitudinal part $\propto \frac{q_\mu q_\nu}{q^2}$, while the self energy is transverse.

The longitudinal part does not affect matrix elements, and is only there because we implicitly chose a gauge by setting  $= -\frac{i g_{\mu\nu}}{q^2} = \frac{-i}{q^2}\left(g_{\mu\nu} - \frac{q_\mu q_\nu}{q^2}\right) - \frac{i}{q^2}\frac{q_\mu q_\nu}{q^2}$.

The only part of the renormalized propagator that does contribute to S-matrix elements is the $g_{\mu\nu}$ part:


$$\longrightarrow \frac{-i g_{\mu\nu}}{q^2(1-\tilde{\Pi}(q^2))}$$

Renormalization Conditions:

The counterterm $\mathcal{L}_{CT} \supset -\frac{A}{4}\tilde{F}_{\mu\nu}\tilde{F}^{\mu\nu}$ contributes to the renormalized photon propagator as,

$$-iAq^2\left(g_{\mu\nu} - \frac{q_\mu q_\nu}{q^2}\right)$$ (w/ external propagators removed)

$$\tilde{\Pi}(q^2) \supset -A$$

We can use this counterterm to <u>fix the residue of the pole in</u>  at $q^2=0$, by choosing

$$\boxed{\tilde{\Pi}(0) = 0}$$

Note that we do not have another counterterm to fix the photon mass to zero, nor do we need one. A consequence of the Ward identity $q_\mu \tilde{\Pi}^{\mu\nu}(q^2) = 0$ has been that the <u>photon remains massless</u> including quantum corrections.

Running of the electric charge

Consider scattering of two electrons:

The effect of renormalizing the photon propagator is to replace

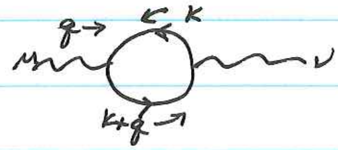
$$\frac{-ig_{\mu\nu}}{q^2} e^2 \to \frac{-ig_{\mu\nu}}{q^2}\left(\frac{e^2}{1-\tilde{\pi}(q^2)}\right)$$

It is as if the electron's charge depends on q^2. In terms of the fine structure constant $\alpha = e^2/4\pi$, we might say,

$$\alpha \to \alpha_{eff}(q^2) = \frac{\alpha}{1-\tilde{\pi}(q^2)}$$

The constant α in the numerator is the fine structure const. when $q^2 = 0$.

Photon Self-Energy: The Calculation

We need to calculate  $\equiv i\Pi^{\mu\nu}(q)$.

$$i\Pi^{\mu\nu}(q) = (-ie)^2\,\overset{\text{from fermion loop}}{(-\mathrm{Tr})}\left[\int \frac{d^4k}{(2\pi)^4}\,\gamma^\mu \frac{i(\not{k}+m)}{k^2-m^2+i\epsilon}\,\gamma^\nu \frac{i(\not{k}+\not{q}+m)}{(k+q)^2-m^2+i\epsilon}\right]$$

Use
$$\mathrm{Tr}\,\gamma^\mu\gamma^\nu = 4g^{\mu\nu}$$
$$\mathrm{Tr}\,\gamma^\mu\gamma^\nu\gamma^\lambda = 0$$
$$\mathrm{Tr}\,\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma = 4(g^{\mu\nu}g^{\rho\sigma} - g^{\mu\rho}g^{\nu\sigma} + g^{\mu\sigma}g^{\nu\rho})$$

$$i\Pi^{\mu\nu}(q) = -e^2\int\frac{d^4k}{(2\pi)^4}\,4\left[\frac{k^\mu(k+q)^\nu - g^{\mu\nu}k\cdot(k+q) + k^\nu(k+q)^\mu + g^{\mu\nu}m^2}{(k^2-m^2+i\epsilon)((k+q)^2-m^2+i\epsilon)}\right]$$

Combine denominators a la Feynman:

$$\left[(k^2-m^2+i\epsilon)((k+q)^2-m^2+i\epsilon)\right]^{-1} = \int_0^1 dx\left[k^2+2x\,k\cdot q + xq^2 - m^2 + i\epsilon\right]^{-2}$$

Complete the square: $\ell^\mu = k^\mu + xq^\mu$

$$\longrightarrow \quad = \int_0^1 dx\left[\ell^2 + x(1-x)q^2 - m^2 + i\epsilon\right]^{-2}$$

Then,

$$i\Pi^{\mu\nu}(q) = -e^2\cdot 4\int_0^1 dx\int\frac{d^4\ell}{(2\pi)^4}\Big\{2\ell^\mu\ell^\nu - g^{\mu\nu}\ell^2 + 2x^2q^\mu q^\nu - 2xq^\mu q^\nu$$
$$+ g^{\mu\nu}(m^2 + xq^2 - x^2q^2)\Big\}\left[\ell^2 + x(1-x)q^2 - m^2 + i\epsilon\right]^{-2}$$

(We have thrown away terms linear in ℓ^μ which integrate to zero.)

Regularization

In order to make divergent integrals that appeared at intermediate stages of our self energy calculations finite, we introduced a hard momentum cutoff $k_E^2 = \Lambda^2$, and took $\Lambda^2 \to \infty$ at the end of the calculation. The hard momentum cutoff violates Lorentz invariance and gauge invariance, and will therefore sometimes lead us astray, even producing nonsensical results.

There are other regularization methods better suited for some theories. As long as the regularization procedure respects the symmetries of the theory, most results will be independent of the regularization method. When they are not, the form of the cutoff must be included as an axiom of the field theory.

Two useful regularization methods are known as regulator fields and dimensional regularization. Here we discuss dimensional regularization, which we will apply to our calculation of the photon self energy.

Dimensional Regularization

Loop integrals are finite in a small enough number of dimensions. Dim Reg is the procedure of analytically continuing integrals in the number of dimensions, and picking out divergences as $d \to 4$.

Consider integrals of the form

$$I = \int \frac{d^d p_E}{(p_E^2+q^2)^n}$$, as might appear after Wick rotating a loop integral.

I is convergent if $n > \frac{d}{2}$, $d = \#$ Euclidean dimensions.

The trick is to turn the denominator into an exponential using the Gamma function,

$$\Gamma(n) = \int_0^\infty dt\, t^{n-1} e^{-t}$$

Change variable $t = \alpha\lambda$, α real > 0.

$$\Gamma(n) = \int_0^\infty d(\alpha\lambda)\, (\alpha\lambda)^{n-1} e^{-\alpha\lambda}$$, i.e.

$$\frac{1}{\alpha^n} = \frac{1}{\Gamma(n)} \int_0^\infty d\lambda\, \lambda^{n-1} e^{-\alpha\lambda}$$

Then we can represent our integral as,

$$I = \frac{1}{\Gamma(n)} \int_0^\infty d\lambda\, \lambda^{n-1} \underbrace{\int d^dK\, e^{-\lambda(K^2+a^2)}}_{e^{-\lambda a^2}\left(\frac{\pi}{\lambda}\right)^{d/2}}$$

$$= \frac{\pi^{d/2}}{\Gamma(n)} \int_0^\infty \lambda^{n-d/2-1}\, e^{-\lambda a^2}\, d\lambda$$

Finally, $\boxed{I = \int \frac{d^dP_E}{(P_E^2+a^2)^n} = \frac{\pi^{d/2}}{\Gamma(n)} \frac{\Gamma(n-d/2)}{a^{2n-d}}}$

'tHooft and Veltman's trick was to analytically continue this formula to arbitrary complex d.

Away from even integers $d \geq 2n$, the Gamma function is well defined.

As $d \to 4$ poles in $(d-4)$ appear, which cancel in convergent combinations.

To <u>renormalize</u> using dim reg, calculate in arbitrary d, include counterterms and impose renormalization conditions, and set $d \to 4$ <u>at the very end.</u> You have to be careful w/ continuing the theory to arbitrary dimensions: the dimensions of couplings depend on d, ie $\frac{e^2}{4\pi} \neq \frac{1}{137}$ in $d \neq 4$.

To pick out the poles as $d \to 4$ use the expansion of the Gamma function about negative integers (or zero):

For $n \geq 0$, $\boxed{\Gamma(-n+\epsilon) = \frac{(-1)^n}{n!}\left[\frac{1}{\epsilon} + \psi(n+1) + O(\epsilon)\right]}$

some finite # (which will not appear in physical results)

$$\psi(1) = -\gamma_E = -0.5772\ldots$$

Euler-Mascheroni constant

<u>Note</u>: It is important to be careful with powers that go to zero as $d \to 4$.

$$A^{4-d} = \exp[(4-d)\log A]$$

$$\approx 1 + \underline{\underline{(4-d)\log A}} + O(4-d)^2$$

If A^{4-d} multiplies a pole from a Γ-function $\propto \frac{1}{4-d}$, then you would miss the finite term $\propto \log A$ if you set $d=4$ too quickly.