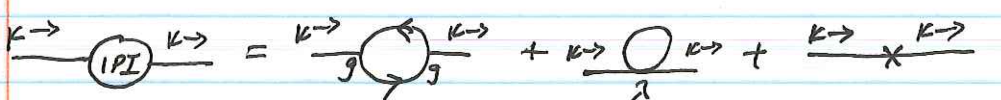
Scalar Self Energy: An Example

Consider the renormalized Lagrangian,

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \tilde{\phi})^2 - \frac{\mu^2}{2}\tilde{\phi}^2 + \bar{\tilde{\psi}}(i\partial\!\!\!/ - m)\tilde{\psi} - g\,\bar{\tilde{\psi}} i\gamma^5 \tilde{\psi}\tilde{\phi} - \frac{\lambda}{4!}\tilde{\phi}^4 + \mathcal{L}_{CT}$$

$$\mathcal{L}_{CT} = \frac{B}{2}(\partial_\mu \tilde{\phi})^2 - \frac{C}{2}\tilde{\phi}^2 + D\,\bar{\tilde{\psi}} i\partial\!\!\!/ \tilde{\psi} - E\,\bar{\tilde{\psi}}\tilde{\psi} - F\,\bar{\tilde{\psi}} i\gamma^5\tilde{\psi}\tilde{\phi} - \frac{G}{4!}\tilde{\phi}^4$$

The scalar self energy to lowest order in the couplings g and λ corresponds to the Feynman diagrams



$+\,\mathcal{O}(g^4, \lambda^2)$

The diagram $-\!\times\!-$ corresponds to the counterterms B and C. We are to think of the counterterms as being of the same order in perturbation theory as to which we are calculating. They will be fixed by the renormalization conditions.

Our first job is to determine the Feynman rule for the counterterms $\xrightarrow{k}\times\xrightarrow{k}$. The only difficulty is the derivatives in $(\partial_\mu\tilde{\phi})^2$. Namely, each ingoing momentum gives an $(-ik^\mu)$ and each outgoing momentum gives a $(+ik_\mu)$ from the derivatives on $a_k e^{-ik\cdot x}$ and $a_k^+ e^{ik\cdot x}$ in $\tilde{\phi}(x)$.

The symmetry in exchanging the two $\tilde{\Phi}$'s cancels the $\frac{1}{2}$ in $\frac{B}{2}$ and $\frac{C}{2}$, giving the naive (and correct) Feynman rule:

$$\boxed{\xrightarrow{k} \times \xrightarrow{k} = i(Bk^2 - C)}$$

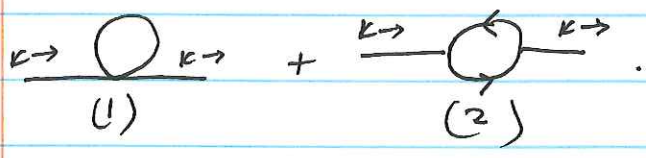
There are two reasons to question this naive treatment of derivatives:

1) Because of the derivative interactions, $\Pi_{\tilde{\Phi}} \equiv \frac{\partial \mathcal{L}}{\partial(\partial_0 \tilde{\Phi})} \neq \partial_0 \tilde{\Phi}$. Hence the canonical commutation relations should be modified.

2) We can't pull derivatives out of the time ordered product: $T(\partial_\mu \phi(x) \cdots) \neq \partial_\mu T(\phi(x) \cdots)$.

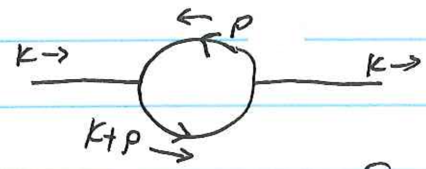
As we will see when we discuss the <u>functional integral formalism</u>, these difficulties are red herrings of the canonical quantization procedure, and you can (often) think of the two problems as cancelling one another.

Next we have to calculate the 1-loop diagrams



(1) is $\frac{(-i\lambda)}{2} \int \frac{d^4p}{(2\pi)^4} \frac{i}{p^2 - \mu^2 + i\epsilon}$. One thing to notice is that (1) is divergent, but it is independent of k, so it can be cancelled by the counterterm C.

Diagram (2) gives a nontrivial contribution to the scalar self energy.

(2) = 

$$= (-ig)^2 \int \frac{d^4p}{(2\pi)^4} (-1)\, \mathrm{Tr}\left[i\gamma^5 \frac{i(\not{p}+m)}{p^2-m^2+i\epsilon}\, i\gamma^5 \frac{i(\not{p}+\not{k}+m)}{(p+k)^2-m^2+i\epsilon} \right]$$

From fermion loop

$$= -g^2 \int \frac{d^4p}{(2\pi)^4} \mathrm{Tr}\left[(\not{p}-m)(\not{p}+\not{k}+m)\right] \frac{1}{p^2-m^2+i\epsilon}\, \frac{1}{(p+k)^2-m^2+i\epsilon}$$

<u>Feynman Parameterization</u>

Feynman is credited with a trick for simplifying loop integrals like this one. The trick is to combine denominators in a clever way.

Consider $\displaystyle\int_0^1 dx \frac{1}{[ax+b(1-x)]^2} = \frac{1}{b-a}\, \frac{1}{ax+b(1-x)}\Big|_0^1$

$$= \frac{1}{b-a}\left(\frac{1}{a}-\frac{1}{b}\right) = \frac{1}{ab}$$

Let $a = (p+k)^2 - m^2 + i\epsilon$, $b = p^2 - m^2 + i\epsilon$

$$\frac{1}{ab} = \int_0^1 dx \frac{1}{[((p+k)^2-m^2+i\epsilon)x + (p^2-m^2+i\epsilon)(1-x)]^2}$$

$$= \int_0^1 dx \left[p^2 + k^2x + 2p\cdot k\, x - m^2 + i\epsilon\right]^{-2}$$

$$= \int_0^1 dx \left[(p+kx)^2 + k^2x(1-x) - m^2 + i\epsilon\right]^{-2}$$

Using this in diagram (2):

$$(2) = -g^2 \int \frac{d^4p}{(2\pi)^4} \mathrm{Tr}\left[(\not{p}-m)(\not{p}+\not{k}+m)\right] \int_0^1 dx \left[(p+kx)^2 + k^2 x(1-x) - m^2 + i\epsilon\right]^{-2}$$

Next we evaluate the Trace: Use $\mathrm{Tr}\,\mathbb{1} = 4$, $\mathrm{Tr}\,\not{p} = 0$, $\mathrm{Tr}\,\not{p}\not{q} = 4p\cdot q$

$$(2) = -g^2 \int \frac{d^4p}{(2\pi)^4} 4(p^2 + p\cdot k - m^2) \int_0^1 dx \left[(p+kx)^2 + k^2 x(1-x) - m^2 + i\epsilon\right]^{-2}$$

To make the integral look more "spherically symmetric" (but remember, these are Lorentz 4-vector dot products) shift the integration variable: $p' \equiv p + kx$

$$(2) = -g^2 \int \frac{d^4p'}{(2\pi)^4} \int_0^1 dx \, \frac{4(p'^2 - 2x p'\cdot k + x^2 k^2 + p'\cdot k - x k^2 - m^2)}{\left[p'^2 + k^2 x(1-x) - m^2 + i\epsilon\right]^2}$$

The terms in the integrand odd in p' vanish when integrated d^4p'. Hence,

$$(2) = -g^2 \int_0^1 dx \int \frac{d^4p'}{(2\pi)^4} \frac{4(p'^2 - (x(1-x)k^2 + m^2))}{\left[p'^2 + k^2 x(1-x) - m^2 + i\epsilon\right]^2}$$

We could evaluate the integral d^4p' if it were in Euclidean space, but it's not. Hence, we rotate the contour of integration over p^0 in order to make the integral spherically symmetric.

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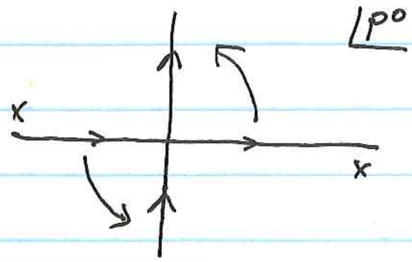
Wick Rotation

Consider integrals of the form,

$$I_n(a) \equiv \int \frac{d^4p}{(2\pi)^4} \frac{1}{(p^2+a)^n} = \int \frac{d^3p\, dp_0}{(2\pi)^4} \frac{1}{(p_0^2-\vec{p}^{\,2}+a)^n}$$

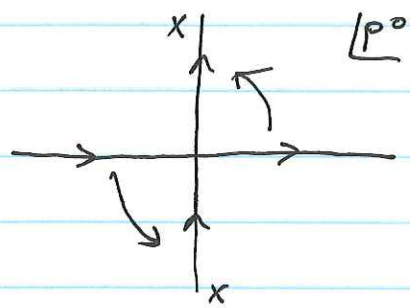
where $\operatorname{Im}(a) > 0$.

If $\operatorname{Re}(\vec{p}^{\,2}-a) > 0$ then the poles in the p^0 plane are like



Rotate contour counterclockwise w/o hitting the poles.

If $\operatorname{Re}(\vec{p}^{\,2}-a) < 0$ then the poles in the p^0 plane are like



Again, rotate counterclockwise to avoid the poles.

After rotating the contour, p^0 runs from $-i\infty$ to $+i\infty$.
Define a new variable $p_4 \equiv -ip_0$.
p_4 runs from $-\infty$ to ∞.
We also have, $dp_0 = i\,dp_4$

$$d^4p = i\,dp_4\, d^3p \equiv i\, d^4p_E \leftarrow \text{Euclidean}$$

Now, $I_n(a) = i\int \frac{d^4 p_E}{(2\pi)^4} \frac{1}{[-p_4^2 - \vec{p}^2 + a]^n}$

$$= i\int \frac{d^4 p_E}{(2\pi)^4} \frac{1}{(-p_E^2 + a)^n}$$

The integral is now 4D spherically symmetric.
Let $z = p_E^2$

$$p_E^3 \, dp_E = \tfrac{1}{2} z \, dz$$

The angular integral is $V(S^3) = 2\pi^2$

$$\left(V(S^d) = \frac{2\pi^{d/2}}{\Gamma(d/2)} \right)$$

Then, $I_n(a) = \frac{i\pi^2}{(2\pi)^4} \int_0^\infty z \, dz \frac{1}{(-z+a)^n}$

$$= \frac{i}{16\pi^2} \frac{(-1)^{n-1}}{(n-1)!} \frac{d^{n-1}}{da^{n-1}} \int_0^\infty dz \frac{z}{-z+a}$$

$$= \frac{(-1)^{n-1}}{(n-1)!} \frac{d^{n-1} I_1(a)}{da^{n-1}}$$

The integral $I_1(a) = \frac{i}{16\pi^2} \int_0^\infty dz \frac{z}{-z+a}$

$$= \frac{i}{16\pi^2} \int_0^\infty dz \left(-1 + \frac{a}{-z+a} \right)$$

is divergent! This is one of those infamous infinities in quantum field theory.

As long as $I_1(a)$ appears as part of a convergent combination,

$$\sum_n c_n I_1(a_n) = \frac{i}{16\pi^2}\int_0^\infty dz \sum_n \frac{c_n z}{-z+a_n} \qquad \text{with} \quad \begin{array}{l} \sum c_n = 0 \\ \sum a_n c_n = 0 \end{array}$$

then we can disregard these fictitious nonphysical divergences. To pick out the divergent parts we can <u>cut off</u> the integral at some large value $z = \Lambda^2$ and at the end let $\Lambda^2 \to \infty$.

$$\sum_n c_n I_1(a_n) = \frac{-i}{16\pi^2}\sum_n \lim_{\Lambda^2\to\infty}\int_0^{\Lambda^2} dz\left(1+\frac{a_n}{z-a_n}\right)c_n$$

$$= -\frac{i}{16\pi^2}\sum_n \lim_{\Lambda^2\to\infty}\left(z + a_n \log|z-a_n|\right)\Big|_{z=0}^{\Lambda^2} c_n$$

$$= \frac{-i}{16\pi^2}\lim_{\Lambda^2\to\infty}\sum_n\Bigg(\underbrace{\Lambda^2 c_n + c_n a_n \log\Lambda^2}_{\sum_n = 0}\Big(1+\underbrace{O\Big(\frac{a_n}{\Lambda^2}\Big)}_{0 \text{ as } \Lambda^2\to\infty}\Big) - c_n a_n \log(-a_n)\Bigg)$$

Hence, in convergent combinations we can take

$$\boxed{I_1(a) = \frac{i}{16\pi^2}\, a \log(-a)}$$

Now $I_2(a)$:

$$I_2(a) = \frac{i}{16\pi^2}\int_0^\infty dz \frac{z}{(z-a)^2}$$

$$= \frac{i}{16\pi^2}\int_0^\infty dz\left(\frac{1}{z-a} + \frac{a}{(z-a)^2}\right)$$

Again consider what happens when $I_2(a)$ is part of a convergent combination:

$$\sum c_n I_2(a_n) = \frac{i}{16\pi^2}\int_0^\infty dz \sum_n \frac{c_n z}{(z-a_n)^2}\ , \quad \sum_n c_n = 0$$

$$= \frac{i}{16\pi^2}\sum_n \lim_{\Lambda^2\to\infty}\int_0^{\Lambda^2} dz \left(\frac{1}{z-a_n} + \frac{a_n}{(z-a_n)^2}\right) c_n$$

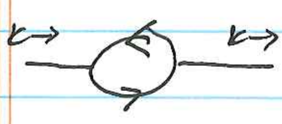
$$= \frac{i}{16\pi^2}\lim_{\Lambda^2\to\infty}\sum_n c_n\Big(\underbrace{\log\Lambda^2}_{\sum_n = 0}\big(1+\underbrace{O\big(\tfrac{a_n}{\Lambda^2}\big)}_{0 \text{ as } \Lambda^2\to\infty}\big) - \underbrace{\frac{a_n}{\Lambda^2}\big(1+O\big(\tfrac{a_n}{\Lambda^2}\big)\big)}_{0 \text{ as } \Lambda^2\to\infty} - \log(-a_n) - \underbrace{1}_{\sum_n = 0}\Big)$$

Hence, $\boxed{I_2(a) = \frac{-i}{16\pi^2}\log(-a)}$ + terms that vanish in convergent combinations

For $n \geq 3$ we can use $I_n(a) = \frac{(-1)^{n-1}}{(n-1)!}\frac{d^{n-1} I_1(a)}{da^{n-1}}$:

$$\boxed{I_n(a) = \frac{i}{16\pi^2}\ \frac{1}{(n-1)(n-2)a^{n-2}}}\ , \quad n \geq 3$$

Back to the self energy: So far we have expressed diagram (2) as:

(2) = 

$$= -g^2\int_0^1 dx \int \frac{d^4p'}{(2\pi)^4}\frac{4\left(p'^2 - (x(1-x)k^2 + m^2)\right)}{[p'^2 + k^2x(1-x) - m^2 + i\epsilon]^2}$$

$$= -g^2\int_0^1 dx \int \frac{d^4p'}{(2\pi)^4}\cdot 4\left[\frac{1}{(p'^2+k^2x(1-x)-m^2+i\epsilon)} - \frac{2(x(1-x)k^2)}{[p'^2+k^2x(1-x)-m^2+i\epsilon]^2}\right]$$

The counterterms B and C will serve to cancel any divergences as the cutoff $\Lambda \to \infty$. Hence we can use our formulae for integrals in convergent combinations.

$$(2) = -4g^2 \int_0^1 dx \Big[I_1\left(k^2 x(1-x) - m^2 + i\epsilon\right)$$

$$- 2x(1-x)k^2 I_2\left(k^2 x(1-x) - m^2 + i\epsilon\right)\Big]$$

$$= -4g^2 \int_0^1 dx \Big[\frac{i}{16\pi^2}\left(k^2 x(1-x) - m^2\right) \log\left(-k^2 x(1-x) + m^2 - i\epsilon\right)$$

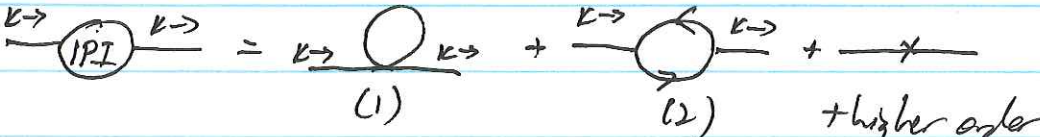
$$+ \frac{i}{16\pi^2} \cdot 2x(1-x)k^2 \log\left(-k^2 x(1-x) + m^2 - i\epsilon\right)\Big]$$

$$\boxed{(2) = -\frac{ig^2}{4\pi^2} \int_0^1 dx \left(3x(1-x)k^2 - m^2\right) \log\left(-k^2 x(1-x) + m^2 - i\epsilon\right)}$$

This is how we will leave the result of the momentum integration. The parameter x is called a <u>Feynman parameter</u>, and we will generalize the technique of reducing ugly Minkowski-space momentum integrals to less ugly integrals over a small number of Feynman parameters.

Summing the 1-loop diagrams including the counterterms, the renormalized 1-loop self energy is,

$-i\tilde{\Pi}(k^2) =$



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$$-i\tilde{\Pi}(k^2) = (1) - \frac{ig^2}{4\pi^2}\int_0^1 dx\,\left(3x(1-x)k^2 - m^2\right)\log\left(-k^2x(1-x)+m^2-i\epsilon\right)$$

$$+Bk^2 - C + (\text{higher order in } g, \lambda) + \text{divergences cancelled by } B, C.$$

The constant diagram (1) can be absorbed in the counterterm C. The counterterms are chosen so that

$$\tilde{\Pi}(\mu^2) = \left.\frac{d\tilde{\Pi}}{dk^2}\right|_{k^2=\mu^2} = 0.$$

If we define,

$$\Pi(k^2) = \frac{g^2}{4\pi^2}\int_0^1 dx\,\left(3x(1-x)k^2 - m^2\right)\log\left(-k^2x(1-x)+m^2-i\epsilon\right)$$

then we can use the counterterms to set

$$\boxed{\tilde{\Pi}(k^2) = \Pi(k^2) - \Pi(\mu^2) - \left.\frac{d\Pi}{dk^2}\right|_{k^2=\mu^2}(k^2-\mu^2)}$$

Finally,

$$\tilde{\Pi}(k^2) = \frac{g^2}{4\pi^2}\int_0^1 dx\left[3x(1-x)k^2 - m^2\right]\log\left(-k^2x(1-x)+m^2-i\epsilon\right)$$

$$-\frac{g^2}{4\pi^2}\int_0^1 dx\left[3x(1-x)\mu^2 - m^2\right]\log\left(-\mu^2x(1-x)+m^2-i\epsilon\right)$$

$$-\frac{g^2}{4\pi^2}\int_0^1 dx\left[3x(1-x)(k^2-\mu^2)\right]\log\left(-\mu^2x(1-x)+m^2-i\epsilon\right)$$

$$-\frac{g^2}{4\pi^2}\int_0^1 dx\,\frac{(3x(1-x)\mu^2-m^2)(-x(1-x)(k^2-\mu^2))}{-\mu^2x(1-x)+m^2-i\epsilon}$$

This is can be simplified:

$$\boxed{\begin{aligned}\tilde{\Pi}(k^2) &= \frac{g^2}{4\pi^2}\int_0^1 dx\,\left(3x(1-x)k^2 - m^2\right)\log\left[\frac{-x(1-x)k^2+m^2-i\epsilon}{-x(1-x)\mu^2+m^2-i\epsilon}\right] \\ &+ \frac{g^2}{4\pi^2}\int_0^1 dx\,\frac{\left(3x(1-x)\mu^2-m^2\right)x(1-x)(k^2-\mu^2)}{m^2-x(1-x)\mu^2-i\epsilon}\end{aligned}}$$

A few comments about this calculation:

1) Notice that we used the counterterms B and C to absorb divergences that appeared at intermediate stages in the calculation. The largest divergence is from the term in the integral $I_1(a)$ proportional to Λ^2. This constant divergence (independent of k^2) is absorbed in the <u>scalar mass counterterm</u> C.

If we think of the scalar + fermion theory as defined by the Lagrangian including the counterterms, then the mass term in the Lagrangian has to be fine tuned by an amount $\mathcal{O}\left(\frac{\mu^2}{\Lambda^2}\right)$ to keep the physical mass^2 at μ^2. This fine tuning is known as the <u>hierarchy problem</u> in the Standard Model, where the <u>Higgs field</u> is a scalar field w/ mass of 125 GeV), and the cutoff Λ is taken to be the Planck scale of $\mathcal{O}(10^{19}\text{ GeV})$.

2) We <u>regularized</u> the momentum integrals by imposing a <u>cutoff</u> on the momentum $p_E^2 = \Lambda^2$, and at the end we took $\Lambda \to \infty$. In a <u>renormalizable</u> field theory physical results are independent of the cutoff as $\Lambda \to \infty$.

Cutoffs in momentum space are not always convenient because they violate symmetries (like Lorentz invariance). We will discuss some of the other ways to regularize divergences in field theory. Physical results should be independent of the method of regularization, as long as it respects the symmetries of the theory.

3) The argument of the log, $-x(1-x)k^2 + m^2 - i\epsilon$, vanishes somewhere in the integration region $x \in [0,1]$ if $k^2 \geq 4m^2$ (since $x(1-x)$ is maximized for $x = \frac{1}{2}$, $x(1-x) = \frac{1}{4}$).

not important here (as $\epsilon \to 0$)

$k^2 = 4m^2$ is the minimum value which allows an intermediate 2-fermion state from $\phi \to \psi + \bar{\psi}$, and is the beginning of a <u>branch cut</u> in $\tilde{\Pi}(k^2)$.

Branch cuts in field theory are commonly associated with intermediate multiparticle states.