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The Higgs Mechanism

Goldstone's theorem taught us that there is a massless particle in the spectrum for every spontaneously broken global symmetry in a quantum field theory. The conclusion is very different for spontaneously broken gauge invariances. Instead, for each spontaneously broken gauge invariance there is a massive vector field.

This fact was first noticed in a nonrelativistic context by Anderson (1958, 1963). The relativistic generalization was described by Brout and Englert (1964), Guralnik, Hagen and Kibble (1964) and Higgs (1964). It is referred to as the Higgs mechanism for short.

The simplest relativistic example of this phenomenon is the Abelian Higgs Model.

Consider QED coupled to a charged scalar field:

$$\mathcal{L} = (D_\mu \phi)^\dagger (D^\mu \phi) + \mu^2 \phi^\dagger \phi - \lambda (\phi^\dagger \phi)^2 - \frac{1}{4} F_{\mu\nu} F^{\mu\nu}$$

$$D_\mu \phi = (\partial_\mu - ieA_\mu)\phi$$

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

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$\mathcal{L}$ is invariant under the gauge transformation

$$\phi(x) \to e^{-i\alpha(x)}\phi(x)$$
$$A_\mu(x) \to A_\mu(x) - \frac{1}{e}\partial_\mu \alpha(x)$$

If $\mu^2 > 0$ and $\lambda > 0$ in the potential $V(\phi) = -\mu^2\phi^\dagger\phi + \lambda(\phi^\dagger\phi)^2$ then $V(\phi)$ is minimized when

$$\phi^\dagger\phi = \frac{v^2}{2} \quad , \quad v = \left(\frac{\mu^2}{\lambda}\right)^{1/2}$$

By a gauge transformation we choose the vacuum expectation value of ϕ to be real. If we write $\phi = \frac{1}{\sqrt{2}}(\phi_1 + i\phi_2)$, ϕ_1 and ϕ_2 real, then

$$\langle 0|\phi_1|0\rangle = v \quad , \quad \langle 0|\phi_2|0\rangle = 0.$$

The choice of vacuum breaks the $U(1)$ gauge invariance. Expanding about the VEV, write the shifted fields

$$\phi_1' = \phi_1 - v \quad , \quad \phi_2' = \phi_2$$

The term $|D_\mu\phi|^2$ in the Lagrangian becomes

$$\begin{aligned} |D_\mu\phi|^2 &= |(\partial_\mu - ieA_\mu)\phi|^2 \\ &= \tfrac{1}{2}(\partial_\mu\phi_1' + eA_\mu\phi_2')^2 + \tfrac{1}{2}(\partial_\mu\phi_2' - eA_\mu\phi_1')^2 \\ &\quad - evA^\mu(\partial_\mu\phi_2' + eA_\mu\phi_1') + \frac{e^2v^2}{2}A^\mu A_\mu \end{aligned}$$

The term $\frac{e^2v^2}{2}A^\mu A_\mu$ looks like a mass term for the gauge field. Mass terms for gauge fields are not gauge invariant, and indeed the gauge invariance of this theory is spontaneously broken by the choice of vacuum, but the full theory maintains gauge invariance. That's good because gauge invariance plays a crucial role in the proof or renormalizability of theories with vector fields.

There is another unusual term in the Lagrangian written in terms of the shifted fields:
$-evA^\mu\partial_\mu\phi_2'$

The field ϕ_2' would be the Goldstone boson in the linear σ-model (i.e. in the absence of the vector field). This can be seen by writing the potential $V(\phi)$ in terms of ϕ_1' and ϕ_2':

$$V(\phi) = \mu^2\phi_1^2 + \ldots$$

mass term for ϕ_1, none for ϕ_2.

The term $-evA^\mu\partial_\mu\phi_2'$ mixes the gauge field with the would-be Goldstone boson, and obscures the physical spectrum of the theory.

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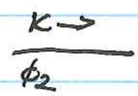
However, we can easily check that the Goldstone-vector mixing term restores the transverseness of the vacuum polarization amplitude, a requirement of gauge invariance.

The Feynman rules for the quadratic terms in the Lagrangian, treated as interactions, are:

$i\, e^2 v^2 g^{\mu\nu}$



$\frac{i}{k^2+i\epsilon}$

$i\, ev(-ik^\mu) = evk^\mu$

The lowest order contributions to the vacuum polarization amplitude are

$$= i\, e^2 v^2 g^{\mu\nu} + (-evk^\mu)\frac{i}{k^2}(evk^\nu)$$

$$= i e^2 v^2 \left(g^{\mu\nu} - \frac{k^\mu k^\nu}{k^2}\right)$$

Transverse, as promised.

Unitary Gauge

The mixing term $evA^\mu\partial_\mu\phi_2'$ obscures the physical degrees of freedom of the theory. To eliminate it we parametrize $\phi(x)$ by polar variables and shift the modulus field:

$$\phi(x) = \frac{1}{\sqrt{2}}\left(v + \eta(x)\right)\exp\left(i\xi(x)/v\right)$$

$$= \frac{1}{\sqrt{2}}\left(v + \eta(x) + i\,\xi(x) + \ldots\right)$$

For small fluctuations $\eta(x) \sim \phi_1'(x)$
$\xi(x) \sim \phi_2'(x)$

The free part of the scalar field Lagrangian is diagonal:

$$\mathcal{L}_0 = \tfrac{1}{2}(\partial_\mu\eta)^2 + \tfrac{1}{2}(\partial_\mu\xi)^2 - \frac{\mu^2}{2}(\eta^2+\xi^2)$$

We now choose a gauge that eliminates $\xi(x)$ from the exponent in $\phi(x)$:

Unitary gauge

$$\begin{cases} \phi^u(x) = \exp(-i\xi(x)/v)\,\phi(x) = \frac{1}{\sqrt{2}}\left(v+\eta(x)\right) \\ B_\mu(x) = A_\mu(x) - \frac{1}{ev}\partial_\mu\xi(x) \end{cases}$$

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In unitary gauge,

$$D_\mu \phi = \exp(-i\xi(x)/v)\left(\partial_\mu \phi^u - ieB_\mu \phi^u\right)$$
$$= \exp(-i\xi(x)/v)\left(\partial_\mu \eta - ieB_\mu (v+\eta)\right)/\sqrt{2}$$

$$|D_\mu \phi|^2 = \frac{1}{2}\left|\partial_\mu \eta - ieB_\mu (v+\eta)\right|^2$$

$$F_{\mu\nu} = \partial_\mu B_\nu - \partial_\nu B_\mu$$

$$\mathcal{L} = \frac{1}{2}\left|\partial_\mu \eta - ieB_\mu (v+\eta)\right|^2 + \frac{\mu^2}{2}(v+\eta)^2 - \frac{\lambda}{4}(v+\eta)^4$$
$$- \frac{1}{4}\left(\partial_\mu B_\nu - \partial_\nu B_\mu\right)^2$$

$$\equiv \mathcal{L}_0 + \mathcal{L}_1$$

$$\mathcal{L}_0 = \frac{1}{2}(\partial_\mu \eta)^2 - \mu^2 \eta^2 - \frac{1}{4}\left(\partial_\mu B_\nu - \partial_\nu B_\mu\right)^2 + \frac{1}{2}e^2 v^2 B_\mu B^\mu$$

$$\mathcal{L}_1 = \frac{1}{2}e^2 B_\mu B^\mu \eta(2v+\eta) - \lambda v^2 \eta^3 - \frac{1}{4}\lambda \eta^4$$

Note that $\xi(x)$ has completely disappeared from the Lagrangian. The would-be Goldstone boson is not in the spectrum. Instead, the vector field B_μ acquires a mass ev. Recall that a massive vector field can have three polarizations, with helicity $\pm 1, 0$. A massless vector can only have helicity ± 1. It is as if the vector field has <u>eaten</u> the Goldstone boson, which becomes the helicity-0 mode of B_μ.

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The Feynman rule for the massive vector field propagator is:

$$\frac{-i\left(g^{\mu\nu} - \frac{k^\mu k^\nu}{m^2}\right)}{k^2 - m^2 + i\epsilon}$$

where m is the mass of the vector field.

The difficulty in proving renormalizability rests in the $k^\mu k^\nu$ term. At large momenta, the $g^{\mu\nu}$ term $\sim \frac{1}{k^2}$, but the $k^\mu k^\nu$ term $\sim$ const. Hence Feynman amplitudes would be more divergent w/ massive vectors if some miraculous cancellations did not occur. In spontaneously broken gauge theories, such cancellations do appear, as proven by 't Hooft (1971).

R_ξ Gauge

Unitary gauge is nice because the physical degrees of freedom are manifest, but renormalizability is difficult to understand. We will now construct a different gauge, for which the physical degrees of freedom and unitarity are not manifest, but propagators fall off at large momenta fast enough for renormalizability to be manifest by naive power counting.

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Recall that by imposing a covariant gauge fixing condition $\partial_\mu A^\mu - f(x) = 0$ and integrating over $f(x)$ against a Gaussian measure, we effectively shifted the Lagrangian by a gauge-fixing term:

$$\mathcal{L}_{g.f.}^{(old)} = -\frac{1}{2\xi}(\partial_\mu A^\mu)^2 \quad , \text{ w/ } \xi \text{ arbitrary.}$$

Instead we could impose the R_ξ gauge fixing condition $\boxed{\partial_\mu A^\mu + \xi m \phi_2 - f(x) = 0}$ and integrate over $f(x)$ against the <u>same</u> Gaussian measure. Here m is the vector boson mass after SSB and $\phi = \frac{1}{\sqrt{2}}(\phi_1 + i\phi_2)$.

The new gauge fixing term is,

$$\boxed{\mathcal{L}_{g.f.} = -\frac{1}{2\xi}\left(\partial_\mu A^\mu + \xi m \phi_2\right)^2}$$

In terms of $\phi_1' = \phi_1 - v$ and $\phi_2' = \phi_2$, the free part of the Lagrangian, including $\mathcal{L}_{g.f.}$, is

$$\mathcal{L}_0 = \frac{1}{2}(\partial_\mu \phi_1')^2 - \mu^2 \phi_1'^2 + \frac{1}{2}(\partial_\mu \phi_2')^2 - \frac{\xi}{2} m^2 \phi_2'^2$$

$$- \frac{1}{4}(\partial_\mu A_\nu - \partial_\nu A_\mu)^2 + \frac{1}{2} m^2 A_\mu A^\mu - \frac{1}{2\xi}(\partial_\mu A^\mu)^2$$

The extra cross-term in $\mathcal{L}_{g.f.}$ cancels the mixing term in the shifted Lagrangian (after integration by parts).

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The Feynman rules for the propagators are:

$$\underset{\phi_1'}{\overset{k\rightarrow}{\text{———}}} \qquad \frac{i}{k^2-2\mu^2+i\epsilon}$$

$$\underset{\phi_2'}{\overset{k\rightarrow}{\text{———}}} \qquad \frac{i}{k^2-\xi m^2+i\epsilon}$$

$$\mu \sim\!\sim\!\sim\!\sim\!\sim \nu \qquad \frac{-i}{k^2-m^2+i\epsilon}\left[g^{\mu\nu}-(1-\xi)\frac{k^\mu k^\nu}{k^2-\xi m^2}\right]$$

$$= -i\left[\frac{g^{\mu\nu}-k^\mu k^\nu/m^2}{k^2-m^2+i\epsilon}+\frac{k^\mu k^\nu/m^2}{k^2-\xi m^2+i\epsilon}\right]$$

In the R_ξ gauge the would-be Goldstone boson ϕ_2' propagates, but the propagator depends on the gauge fixing parameter ξ.

But now the vector boson propagator falls like $\frac{1}{k^2}$ at large momentum, so the high energy behavior is manifestly more mild than in unitary gauge.

(Fujikawa, Lee, Sanda (1972); Yao (1973))

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In QED in covariant gauges the Fadeev-Popov determinant is independent of the fields and can be factored out of the functional integral. However, in R_ξ gauge it's not so simple.

The gauge fixing condition is

$$G(A_\mu \phi) = \partial_\mu A^\mu + \xi m \phi_2 - f(x) = 0$$

In terms of the shifted fields $\phi_1' = \phi_1 - v$, $\phi_2' = \phi_2$, the gauge invariance is (infinitessimally)

$$\begin{aligned}
\phi_1' &\to \phi_1'(x) - \alpha(x)\,\phi_2'(x) \\
\phi_2' &\to \phi_2' + \alpha(x)\,(v + \phi_1') \\
A_\mu &\to A_\mu - \frac{1}{e}\partial_\mu \alpha(x)
\end{aligned}$$

The Fadeev-Popov determinant is

$$\det\left(\frac{\delta G}{\delta \alpha}\right) = \det\left(-\frac{1}{e}\partial_\mu\partial^\mu + \xi m (v + \phi_1')\right)$$

Introducing Fadeev-Popov ghost fields $c(x), \bar{c}(x)$, we can absorb the determinant into a functional integral over the ghosts, with ghost Lagrangian

$$\boxed{\mathcal{L}^{R_\xi}_{ghost} = \bar{c}\left(-\partial_\mu\partial^\mu + \xi m^2\left(1 + \frac{\phi_1'}{v}\right)\right)c}$$

ghost coupling to physical scalar.

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The non-Abelian generalization of the Higgs mechanism is straightforward, as is the discussion of unitary gauge and R_ξ-gauge. (See Peskin + Schroder Ch. 20.1 and 21.1, Cheng + Li Ch. 8.3, 9.2)

In the Standard Model, the <u>Higgs doublet</u> is an $SU(2)_W$ doublet and is charged under $U(1)_Y$.

Weak gauge invariance — Hypercharge gauge invariance

The Higgs field acquires a vacuum expectation value, which spontaneously breaks $SU(2)_W \times U(1)_Y$ to $U(1)_{EM}$.

Electromagnetism

That is why, if the Standard Model is correct, the $W^\pm$ and Z bosons are heavy, while the photon is massless.