1/9

Spontaneous Symmetry Breaking

Consider a theory of N real scalar fields ϕ_i with Lagrangian

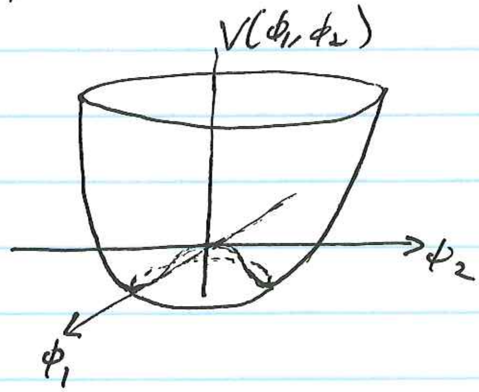
$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi_i)^2 + \frac{1}{2}\mu^2 \phi_i^2 - \frac{\lambda}{4}(\phi_i^2)^2 \qquad \text{(\underline{Linear } \sigma\text{\underline{-model}})}$$

where $\phi_i^2 \equiv \sum_i \phi_i^2$, $(\partial_\mu \phi_i)^2 \equiv \sum_i \partial_\mu \phi_i \partial^\mu \phi_i$

$\mathcal{L}$ is invariant under the $O(N)$ symmetry

$$\phi_i \to R_{ij}\, \phi_j \quad , \quad R = N \times N \text{ orthogonal matrix}$$

Notice that the mass term $\frac{1}{2}\mu^2\phi_i^2$ has the opposite sign to that we expect for a field w/ mass μ. If $\mu^2 > 0$, $\lambda > 0$, and we write $\mathcal{L} = \frac{1}{2}(\partial_\mu \phi_i)^2 - V(\phi_i)$, then the potential $V(\phi_i)$ has the following shape:



$(N=2)$

The potential is minimized for any ϕ_0^i satisfying

$$\boxed{(\phi_0^i)^2 = \frac{\mu^2}{\lambda}}$$

The continuous set of vacua are related by an $O(N)$ transformation, and are physically equivalent. We can choose:

$$\boxed{\phi_0^i = (0,0,\dots,v), \quad v = \frac{\mu}{\sqrt{\lambda}}}$$

Expanding the fields about the classical vacuum, we write

$$\phi_i(x) = (\pi_j(x), v+\sigma(x)) \qquad j=1,\dots,N-1$$

In terms of the shifted fields $\pi_j(x)$, $\sigma(x)$, the Lagrangian is:

$$\mathcal{L} = \frac{1}{2}\sum_{j=1}^{N-1}(\partial_\mu \pi_j)^2 + \frac{1}{2}(\partial_\mu\sigma)^2 - \mu^2\sigma^2$$

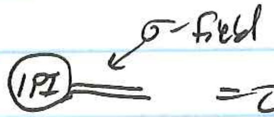
$$-\sqrt{\lambda}\mu\sigma^3 - \sqrt{\lambda}\mu(\pi_j)^2\sigma - \frac{\lambda}{4}\sigma^4 - \frac{\lambda}{2}(\pi_j)^2\sigma^2 - \frac{\lambda}{4}((\pi_j)^2)^2$$

The fluctuations about the vacuum describe $(N-1)$ massless fields π_j, and a massive field σ with mass $\boxed{m_\sigma = \sqrt{2}\mu}$.

The σ field describes fluctuations in the radial direction ("up" the potential); The π_j fields describe fluctuations along the trough of the potential.

An $O(N-1)$ symmetry is manifest — this is the symmetry of the classical vacuum. The $O(N)$ sym is broken to $O(N-1)$.

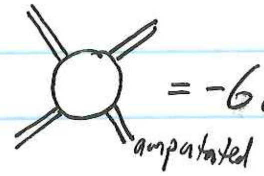
The discussion so far has been completely classical. The parameters in the Lagrangian are renormalized. Luckily, despite the spontaneous symmetry breaking in the theory, it is often the case (as it is in this theory) that the regularization procedure can be chosen to preserve the full symmetry of the theory (in this case $O(N)$). In other words, only 3 renormalization conditions are required in the linear σ-model: The field renormalization, μ^2 and λ need to be fixed by renormalization conditions. In the σ-model one choice is



$= 0$

$\frac{d}{dp^2}\Big($ 

$\Big)\Big|_{p^2=m_\sigma^2} = 0$



$= -6i\lambda$ at $s = 4m^2$, $t = u = 0$.

The Effective Action

It is possible to construct a function in the QFT that determines the exact relationship between the vacuum expectation value of the fields $\langle \phi_i \rangle$ and the parameters in the Lagrangian μ^2 and λ.

4/9

The construction is similar to the construction of the Gibbs free energy in Statistical Mechanics.

Define the <u>generating functional</u> of correlation functions:

$$Z[J(x)] \equiv \int_{t=0}^{T} \mathcal{D}\phi(x)\, \exp\left[i\int d^4x \left(\mathcal{L}[\phi] + J(x)\,\phi(x)\right)\right]$$

$$= \langle 0 | e^{-iHT} | 0 \rangle_{J(x)}$$

$$\equiv e^{-iE[J]}$$

Up to a factor of T, $E[J]$ is the vacuum energy in the presence of the <u>source</u> $J(x)$.

$Z[J(x)]$ encodes expectation values of time ordered products of fields:

$$\frac{1}{Z[J]}\frac{\delta^n Z[J]}{\delta J(x_1)\cdots\delta J(x_n)} = \frac{i^n \int \mathcal{D}\phi\, e^{i\int d^4x(\mathcal{L}+J\phi)}\,\phi(x_1)\cdots\phi(x_n)}{\int \mathcal{D}\phi\, e^{i\int d^4x(\mathcal{L}+J\phi)}}$$

Setting $J=0$ then gives $i^n \langle 0 | T\phi(x_1)\cdots\phi(x_n)|0\rangle$ in the theory w/o the source.

In the presence of the source, we define

$$\frac{\int \mathcal{D}\phi\, \exp[i\int d^4x(\mathcal{L}+J\phi)]\,\phi(x_1)\cdots\phi(x_n)}{\int\mathcal{D}\phi\,\exp[i\int d^4x(\mathcal{L}+J\phi)]} = \langle 0 | T\phi(x_1)\cdots\phi(x_n)|0\rangle_J$$

Varying $Z[J]$ with respect to a single source $J(x)$,

$$\frac{\delta E[J]}{\delta J(x)} = i\,\frac{1}{Z[J]}\,\frac{\delta Z[J]}{\delta J(x)} = -\langle 0|\phi(x)|0\rangle_J$$

$$\equiv -\phi_{cl}(x)$$

The effective action is defined by a Legendre transform of $E[J]$:

$$\boxed{\Gamma[\phi_{cl}] \equiv -E[J] - \int d^4x\, J(x)\,\phi_{cl}(x)}$$ Effective Action

Varying $\Gamma[\phi_{cl}]$ with respect to ϕ_{cl} gives the source J:

$$\boxed{\frac{\delta \Gamma[\phi_{cl}]}{\delta \phi_{cl}(x)}} = \frac{-\delta}{\delta\phi_{cl}(x)} E[J] - \int d^4y\, \frac{\delta J(y)}{\delta \phi_{cl}(x)}\,\phi_{cl}(y) - J(x)$$

$$= -\int d^4y\, \frac{\delta J(y)}{\delta\phi_{cl}(x)} \underbrace{\frac{\delta E[J]}{\delta J(y)}}_{-\phi_{cl}(y)} - \int d^4y\, \frac{\delta J(y)}{\delta\phi_{cl}(x)}\,\phi_{cl}(y) - J(x)$$

$$\boxed{= -J(x)}$$

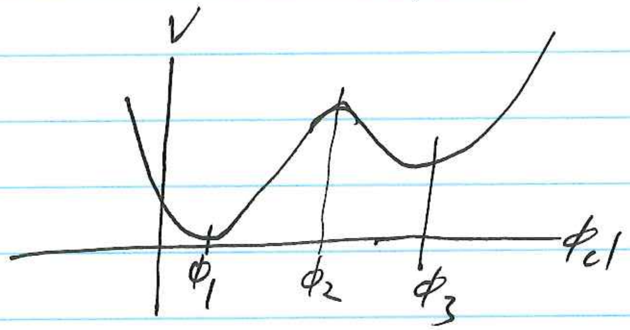
In the absence of the source $J(x)$, the vacuum expectation value ϕ_{cl} satisfies $\boxed{\dfrac{\delta \Gamma[\phi_{cl}]}{\delta\phi_{cl}(x)} = 0}$ Determines ϕ_{cl}

If the vacuum is Lorentz invariant, then $\Gamma[\phi_{cl}]$ is proportional to the volume of spacetime VT, so we can define the <u>effective potential</u>

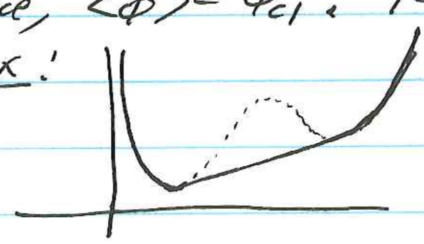
$$\Gamma[\phi_{cl}] = -(VT)\, V_{eff}(\phi_{cl})$$

In the ground state, $\boxed{\dfrac{\partial V_{eff}(\phi_{cl})}{\partial \phi_{cl}} = 0}$

We have assumed that the ground state has a uniform value of ϕ everywhere. In the presence of local minima in the classical potential this need not be true. Consider an effective potential like the following:



For a ϕ_{cl} between the local minima ϕ_1, ϕ_2, the energy can be minimized by forming domains with macroscopic regions of $\langle\phi\rangle = \phi_1$ and $\langle\phi\rangle = \phi_3$, such that averaged over all space, $\langle\phi\rangle = \phi_{cl}$. The resulting effective potential is <u>convex</u>!



7/9

Goldstone's Theorem

In the $O(N)$ linear σ-model studied above there were $N-1$ massless fields that we called $\pi_j(x)$. These fields remain massless in the full quantum theory. This is a reflection of Goldstone's Theorem:

> For every spontaneously broken continuous symmetry the theory contains a massless particle.

In the $O(N)$ linear σ-model, the $O(N)$ symmetry of the theory is spontaneously broken to $O(N-1)$.

The number of generators of $O(N)$ is $\frac{N(N-1)}{2}$, which can be thought of as rotations in any of $\frac{N(N-1)}{2}$ planes.

The number of broken generators is $\frac{N(N-1)}{2} - \frac{(N-1)(N-2)}{2} = \underline{\underline{N-1}}$.

Hence we expect $N-1$ massless Goldstone bosons, the π_j fields.

also called Nambu-Goldstone bosons.

Current algebra techniques provide a simple proof of Goldstone's theorem.

8/9

Suppose there is a continuous symmetry in a theory. From Nöether's theorem there is a conserved current $J^\mu(x)$ with $\partial_\mu J^\mu = 0$.

Suppose the fields transform as,

$$\phi(x) \to \phi'(x) = e^{i\theta^a Q^a} \phi(x) e^{-i\theta^b Q^b}$$
$$= \phi(x) + i\theta^a [Q^a, \phi] + \mathcal{O}(\theta^a)^2$$

where Q^a are the charges and θ^a are the parameters of the symmetry transformation.

Current conservation implies

$$0 = \int d^3x \, [\partial^\mu J^a_\mu(\vec{x}, t), \phi(0)]$$

$$= \partial^0 \int d^3x \, [J^a_0(\vec{x}, t), \phi(0)] + \text{surface term}$$

$$= \frac{d}{dt} [Q^a(t), \phi(0)] + 0.$$

The signal of spontaneous symmetry breaking is the nontrivial action of the symmetry on the vacuum,
$Q^a(t)|0\rangle \neq 0$.

Equivalently, $\boxed{\langle 0 | [Q^a(t), \phi(0)] | 0 \rangle \equiv \eta^a \neq 0}$ condition for SSB for some set $\{a\}$

Now insert a complete set of states, use translation operator:

$$\eta = \sum_n (2\pi)^3 \delta^3(\vec{p}_n) \left(\langle 0 | J_0(0) | n \rangle \langle n | \phi(0) | 0 \rangle e^{-iE_n t} - \langle 0 | \phi(0) | n \rangle \langle n | J_0(0) | 0 \rangle e^{iE_n t} \right)$$

η is nonvanishing and time independent.
The delta function fixes $\vec{p}_n = 0$. If $E_n \neq 0$ when $\vec{p}_n = 0$, then the two terms, w/ $e^{-iE_n t}$ and $e^{+iE_n t}$, would not cancel.

Hence, for each generator such that $Q^a|0\rangle \neq 0$, there must be an intermediate state for which
$\boxed{E_n = 0 \text{ when } \vec{p}_n = 0}$ → massless Goldstone boson

The particle is such that $\boxed{\begin{array}{l}\langle n|\phi(0)|0\rangle \neq 0 \\ \text{and } \langle n|J_0^a(0)|0\rangle \neq 0\end{array}}$
where $a \in$ set of labels of broken generators.

This argument is completely general, and did not rely on perturbation theory.

Note that broken <u>gauge invariances</u> do not necessarily lead to massless particles. Instead, gauge bosons become massive. This is the story of the Higgs mechanism.

Spontaneously broken <u>approximate</u> symmetries lead to nearly massless particles.
The pions π^+, π^-, π^0 are the almost-Goldstone bosons due to the breaking of the $SU(2)_L \times SU(2)_R$ chiral symmetry of the light up, down quarks to $SU(2)$-isospin.