Functional Integral Quantization of the Electromagnetic Field

Electrodynamics is especially difficult to quantize by the canonical quantization procedure because of gauge invariance, so until now we have not derived the Feynman rules for QED, but only motivated them by analogy with the scalar field.

The functional integral formalism makes it reasonably straightforward to sort through the complications of gauge invariance, and will provide an easy derivation of the Ward Identities.

Consider the free EM field, with action

$$\begin{aligned} S &= \int d^4x \left(-\tfrac{1}{4} F_{\mu\nu} F^{\mu\nu}\right) \\ &= \int d^4x \ \tfrac{1}{2} A_\mu(x) \left(g^{\mu\nu}\partial^2 - \partial^\mu\partial^\nu\right) A_\nu(x) \\ &= \int \frac{d^4k}{(2\pi)^4} \tfrac{1}{2} \widehat{A}_\mu(k) \left(-k^2 g^{\mu\nu} + k^\mu k^\nu\right) \widehat{A}_\nu(-k) \end{aligned}$$

If we naively apply our rules to calculate the photon propagator, we get

$$\begin{aligned} \langle 0 | T[A_\mu(x) A_\nu(y)] | 0 \rangle &= \frac{\int \prod_\alpha \mathcal{D}A^\alpha \ A_\mu(x) A_\nu(y)\, e^{iS}}{\int \prod_\beta \mathcal{D}A^\beta \ e^{iS}} \\ &= i\left(g^{\mu\nu}\partial^2 - \partial^\mu\partial^\nu\right)^{-1}. \end{aligned}$$

However, the inverse is not defined because the differential operator $(g^{\mu\nu}\partial^2 - \partial^\mu\partial^\nu)$ is singular.

The mistake we made was in integrating over all four components of the gauge field.

Gauge invariance: $A_\mu(x) \to A_\mu(x) + \frac{1}{e}\partial_\mu \alpha(x)$

- Implies that a continuum of field configurations are physically equivalent.

By integrating over all four components of $A^\mu(x)$ we are redundantly counting each independent field configuration an infinite number of times.

Gauge Fixing

To fix the problem, Fadeev and Popov had the idea to include a gauge fixing condition via a δ-function in the functional integral.

Say we want to fix a gauge such that some function $G(A^\mu) = 0$. For example, $G(A^\mu) = \partial_\mu A^\mu \to$ Lorenz gauge

$G(A^\mu) = \nabla \cdot \vec{A} \to$ Radiation gauge/Coulomb gauge

we insert into the functional integral

$$\boxed{1 = \int \mathcal{D}\alpha(x) \; \delta\left(G(A_\mu^{(\alpha)})\right) \det\left(\frac{\delta\left(G(A_\mu^{(\alpha)})\right)}{\delta\alpha}\right)}$$

where $A_\mu^{(\alpha)} = A_\mu + \frac{1}{e}\partial_\mu\alpha$ is the gauge-transformed A_μ

This is the infinite-dimensional generalization of

$$\delta(g(x)) = \frac{1}{|g'(x_0)|}\delta(x-x_0), \text{ where } g(x_0)=0$$

(if there is a unique x_0 s.t. $g(x_0)=0$.)

The δ-function allows us to identify the redundancy due to integration over equivalent field configurations.

$$\int \prod_\beta \mathcal{D}A^\beta \, e^{iS} \, \mathcal{O}(A^\mu)$$

← Any gauge-invariant operator

$$= \int \mathcal{D}\alpha \int \prod_\beta \mathcal{D}A^\beta \, e^{iS[A^\mu]} \det\left(\frac{\delta G(A_\mu^{(\alpha)})}{\delta \alpha}\right) \mathcal{O}(A^\mu) \, \delta\left(G(A^{(\alpha)})\right).$$

Changing variables to $A_\mu^{(\alpha)}$ and renaming it A_μ,

$$\int \prod_\beta \mathcal{D}A^\beta e^{iS[A^\mu]} \mathcal{O}(A^\mu) = \int \mathcal{D}\alpha \int \mathcal{D}A \, e^{iS[A]} \, \delta\left(G(A)\right) \mathcal{O}(A) \det\left(\frac{\delta G}{\delta \alpha}\right).$$

The integral over $\mathcal{D}\alpha$ represents the redundant integration over gauge-equivalent field configurations.

<u>Example</u>: Generalization of Lorenz gauge

$$G(A_\mu) = \partial^\mu A_\mu - f(x), \quad \text{with } f(x) \text{ some function.}$$

Gauge transformation: $A_\mu \to A_\mu + \frac{1}{e}\partial_\mu \alpha$

$$\delta G(A_\mu) = \frac{1}{e}\partial_\mu \partial^\mu \alpha$$

$$\det\left(\frac{\delta G}{\delta \alpha}\right) = \det\left(\frac{1}{e}\partial^2\right)$$

<u>Note</u>: With this choice of $G(A)$ the determinant is independent of A_μ. Otherwise, Fadeev and Popov's trick is to introduce fictitious fields (ghosts) s.t. the Gaussian functional integral over those fields reproduces the determinant. We will see how this works a bit later.

So, $\int \prod_\mu \mathcal{D}A^\mu \, e^{iS[A^\mu]} \, \mathcal{O}(A^\mu)$

$$= \int \mathcal{D}\alpha \int \prod_\mu \mathcal{D}A^\mu \, e^{iS[A^\mu]} \mathcal{O}(A^\mu) \, \delta(\partial^\mu A_\mu - f(x)) \det\left(\tfrac{1}{e}\partial^2\right)$$

The determinant and the $\int \mathcal{D}\alpha$ just give an infinite constant that factors out of physical quantities.

We could have chosen any $f(x)$, and gotten the same result (for gauge-invariant observables). We could also integrate over functions $f(x)$ with an appropriate normalization that factors out of physical calculations anyway:

$$\int \prod \mathcal{D}A \, e^{iS[A]} \mathcal{O}(A)$$
$$= \Big(\underset{\uparrow}{N(\xi)} \det\left(\tfrac{1}{e}\partial^2\right) \int \mathcal{D}\alpha \Big) \int \mathcal{D}f(x) \exp\left[-i \int d^4x \, \frac{f(x)^2}{2\xi}\right]$$
normalization
$$\times \int \prod_\mu \mathcal{D}A^\mu \, e^{iS[A]} \mathcal{O}(A) \, \delta(\partial^\mu A_\mu - f(x))$$

$$= \Big(N(\xi) \det\left(\tfrac{1}{e}\partial^2\right) \int \mathcal{D}\alpha \Big) \int \prod_\mu \mathcal{D}A^\mu \, e^{iS[A]} \mathcal{O}(A) \exp\left[-i \int d^4x \, \frac{(\partial^\mu A_\mu)^2}{2\xi}\right],$$

where we have done the δ-function integral over $f(x)$, and ξ = any finite constant.

The effect of integrating over gauge-equivalent field configurations can be remedied by adding to the action a term $-\int d^4x \, \frac{(\partial^\mu A_\mu)^2}{2\xi}$.

Physical results should be independent of the arbitrary ξ.

We can now calculate correlators:

$$\boxed{\langle 0|T[\mathcal{O}(A)]|0\rangle = \lim_{T\to\infty(1-i\epsilon)} \frac{\int \mathcal{D}A\, \mathcal{O}(A) \exp\left[i\int_{-T}^{T} d^4x \left(\mathcal{L} - \frac{1}{2\xi}(\partial_\mu A^\mu)^2\right)\right]}{\int \mathcal{D}A\, \exp\left[i\int_{-T}^{T} d^4x \left(\mathcal{L} - \frac{1}{2\xi}(\partial_\mu A^\mu)^2\right)\right]}}$$

All of the ugly infinite constant factors cancelled, leaving only the ξ-dependent contribution to the Lagrangian.

The gauge-fixed action is

$$S_{gf} = \frac{1}{2}\int d^4x\, d^4x'\, A_\mu(x) \underbrace{\delta^4(x-x')\left(g^{\mu\nu}\partial^2 - (1-\tfrac{1}{\xi})\partial^\mu\partial^\nu\right)}_{i K^{\mu\nu}(x,x')} A_\nu(x')$$

The Kernel whose inverse is the photon propagator is

$$K^{\mu\nu}(x,x') = -i\,\delta^4(x-x')\left(g^{\mu\nu}\partial^2 - \partial^\mu\partial^\nu + \tfrac{1}{\xi}\partial^\mu\partial^\nu\right)$$

$$= -i\,\delta^4(x-x')\left[\underbrace{\left(g^{\mu\nu} - \frac{\partial^\mu\partial^\nu}{\partial^2}\right)}_{P^{\mu\nu}_{(T)}}\partial^2 + \frac{1}{\xi}\underbrace{\left(\frac{\partial^\mu\partial^\nu}{\partial^2}\right)}_{P^{\mu\nu}_{(L)}}\partial^2\right]$$

$P^{\mu\nu}_{(T)}$ transverse projector

$P^{\mu\nu}_{(L)}$ Longitudinal projector

$$P_{(T)}^2 = P_{(T)}, \quad P_{(L)}^2 = P_{(L)}, \quad P_{(T)}{}^\mu{}_\nu + P_{(L)}{}^\mu{}_\nu = \delta^\mu{}_\nu.$$

To invert the Kernel, we just invert the coefficients of the projection operators.

$$\boxed{``\langle 0|T[A^\mu(x)A^\nu(y)]|0\rangle" = \int \frac{d^4k}{(2\pi)^4}\frac{-i e^{-ik\cdot(x-y)}}{k^2+i\epsilon}\left[\left(g^{\mu\nu} - \frac{k^\mu k^\nu}{k^2}\right) + \xi\frac{k^\mu k^\nu}{k^2}\right]}$$

Exercise: check that $K^{\mu\nu}(x,x')K^{-1}_{\mu\nu}(x',x'') = \delta^4(x-x'')$.

We have now derived the Feynman rule for the photon propagator:

$$\boxed{\sim\sim \quad \frac{-i}{k^2+i\epsilon}\left(g^{\mu\nu} - (1-\xi)\frac{k^\mu k^\nu}{k^2}\right)}$$

Special choices of ξ:

$\xi=0$: $\sim\sim = \frac{-i}{k^2+i\epsilon}\left(g^{\mu\nu} - \frac{k^\mu k^\nu}{k^2}\right)$ Landau gauge.

$\xi=1$: $\sim\sim = \frac{-i}{k^2+i\epsilon} g^{\mu\nu}$ Feynman gauge.

$\xi=-3$: $\sim\sim = \frac{-i}{k^2+i\epsilon}\left(g^{\mu\nu} - 4\frac{k^\mu k^\nu}{k^2}\right)$ Traceless gauge.